MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: On computing flat outputs using Pfaffian systems

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Academic year: 2013-2014

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1 Introduction

Feedback linearization of control systems allows us to apply the theory of linear systems to the nonlinear ones and to design inputs in order to move the system along a trajectory given initial and final points.

A particular case of (dynamic) feedback linearization is to linearize using the Goursat normal form. Once the Goursat normal form is found, the flat outputs are derived easily. This procedure requires several computations to determine if a system can be linearizable by feedback linearization. However, for nonholonomic systems, it becomes an easier task.

The compilation of results involving feedback linearization and the computation of flat outputs using Pfaffian systems are the focus of our work.

This project is divided into 5 topics. First of all we give the algebraic notions and several results involving exterior differential systems that will be used through the different chapters as well as the theory about Goursat normal forms and how to obtain them. All this is contained in chapters 2 to 5.

In chapter 6, we will focus on systems with m inputs and $m+2$ variables, where $m \geq 4$. Here we give different procedures to put the forms into a Goursat normal form depending on the parity of the system dimension. These algorithms can be applied to systems which have, at least, one form of rank greater than 1. The case of 5 variables and 3 inputs is treated separatedly in the same chapter.

An algorithm to linearize systems by means of a diffeomorphism and a static feedback using one forms is outlined in chapter 7. It is the exterior differential counterpart of the well-kwon procedure that uses vector flieds.

Some results about dynamic feedback linearization using the notion of dynamic immesrion are presented in the last chapter. A necessary and sufficient condition for driftless systems with two inputs is stated and proven.

2
Algebraic notions

2.1 Multilinear algebra and ideals

2.1.1 Definition An *algebra*$(V, \odot)$, is a vectorial space V over a field (we will normally use the real field), with a multiplicative operation $\odot : V \times V \longrightarrow V$ that satisfies:

- Given an scalar $\alpha \in \mathbb{R}$, $\alpha(a \odot b) = (\alpha a) \odot b = a \odot (\alpha b)$.
- If there exists an element $e \in V$ such that $x \odot e = e \odot x = x$, $\forall x \in V$, then it is unique and we call it *neutral* or *identity element*.

2.1.2 Definition Let $(V, \odot)$ be an algebra, we say that a subspace $W \subset V$ is an *algebraic ideal* if:

$$x \in W, \quad y \in V \Longrightarrow x \odot y, y \odot x \in W$$

We recall that the intersection of ideals is also an ideal.

2.1.3 Definition Let $(V, \odot)$ be an algebra and let $A := \{a_i \in V, \quad 1 \leq i \leq K\}$ be any finite collection of linearly independent elements in V. Let S be the set of all ideals containing A, i.e:

$$S := \{I \subset V, I \text{ ideal}, A \subset I\}$$

The ideal I_A generated by A is defined as:

$$I_A = \bigcap_{I \in S} I$$

and is the minimal ideal in S containing A. We call it *minimal ideal*.

2.1.4 Theorem *Let $(V, \odot)$ be an algebra with an identity element. Let $A := \{a_i \in V, \quad 1 \leq i \leq K\}$ be a finite collection of elements in V and I_A the ideal generated by A. Then for each $x \in I_A$ there exist vectors $v_1, \ldots, v_K \in V$ such that*

$$x = v_1 \odot a_1 + v_2 \odot a_2 + \ldots + v_K \odot a_K$$

2.1.5 Definition Let $(V, \odot)$ be an algebra and $I \subset V$ an ideal. Two vectors $x, y \in V$ are said to be *equivalent modulus* I if and only if $x - y \in I$. This equivalence is denoted by

$$x \equiv y \mod I$$

If the space $(V, \odot)$ has an identity element, the above definition implies that there exists equivalence between vectors if and only if

$$x - y = \sum_{i=1}^{K} \theta_i \odot \alpha_i$$

for any $\theta_K \in V$. We will denote it as

$$x \equiv y \mod \alpha_1, \alpha_2, \ldots, \alpha_K$$

due to the fact that the modulus operation is performed over the ideal generated by $\alpha_1, \alpha_2, \ldots, \alpha_K$.

2.2 Exterior algebra

We consider V a vectorial space, V^* its dual space and $\Lambda^k(V^*)$ the vectorial space of the alternating k-tensors with a multiplicative operation.
The wedge product is the usual operation but this operation is not closed in the space $\Lambda^k(V^*)$. Therefore, $\Lambda^k(V^*)$ is not an algebra with this operation.
We define the direct sum operation on the all alternating tensors space as:

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^n(V^*)$$

Then, given $\xi \in \Lambda(V^*)$, this tensor can be writen as $\xi = \xi_0 + \xi_1 + \cdots + \xi_n$ where each $\xi_p \in \Lambda^p(V^*)$.

Notice that $\Lambda(V^*)$ is closed under the exterior multiplication. It is therefore an algebra.

2.2.1 Definition The space of all the alternating tensors with the exterior product, $(\Lambda(V^*), \wedge)$, is an algebra, called the *exterior algebra* over V^*.

We note that the algebra $(\Lambda(V^*), \wedge)$ has the identity element since $1 \in \Lambda^0(V^*)$. The theorem (2.1.4) implies that the ideal generated by a finite set

$$\Sigma := \{\alpha_i \in \Lambda(V^*), \quad 1 \leq i \leq K\}$$

can be written as

$$I_\Sigma = \left\{ \pi \in \Lambda(V^*) \,|\, \pi = \sum_{i=1}^{K} \theta_i \wedge \alpha_i, \quad \theta_i \in \Lambda(V^*) \right\}$$

Given an arbitrary set Σ, of linearly independent generators, it may also be possible to generate I_Σ with a smaller set of generators Σ'.

2.3 Systems of Exterior equations

The goal of this section is to solve the following system of equations

$$\alpha_1 = 0, \dots, \alpha_K = 0$$

where $\alpha_i \in \Lambda(V^*)$.

2.3.1 Definition A *system of exterior equations* over V is a finite set of linearly independent equations

$$\alpha_1 = 0, \dots, \alpha_K = 0$$

where each $\alpha_i \in \Lambda^k(V^*)$ for some $1 \leq k \leq n$. A solution to a system of exterior equations is any subspace $W \subset V$ such that

$$\alpha_1|_W \equiv 0, \dots, \alpha_K|_W \equiv 0$$

where $\alpha|_W$ stands for $\alpha(v_1, \dots, v_k)$ for all $v_1, \dots, v_k \in W$.

We have to keep in mind that there is not uniqueness of the solutions of this system since any subspace $W_1 \subset W$ satisfies $\alpha|_{W_1} \equiv 0$ if $\alpha|_W \equiv 0$.

2.3.2 Theorem *Given a system of exterior equations $\alpha_1 = 0, \dots, \alpha_K = 0$, and the corresponding I_Σ generated by the collection of alternating tensors $\Sigma := \{\alpha_1, \dots, \alpha_K\}$ where $\alpha_i \in \Lambda(V^*)$. A subspace W solves the system of exterior equations if and only if also satisfies $\pi\,|_W \equiv 0$ for all $\pi \in I_\Sigma$.*

Proof:
If $\pi|_W \equiv 0$ for all $\pi \in I_\Sigma$ then, since the ideal is generated by $\Sigma = \{\alpha_1, \dots \alpha_K\}$, each α_i belong in I_Σ and consequently $\alpha_i|W \equiv 0$, $\forall \alpha_i \in I_\Sigma$.

Reciprocally, if $\pi \in I_\Sigma$, it can be writen as

$$\pi = \sum_{i=1}^{K} \theta_i \wedge \alpha_i, \quad \theta_i \in \Lambda(V^*)$$

Hence, if $\alpha_i\,|_W \equiv 0$ for $1 \leq i \leq K$ implies that $\pi|_W \equiv 0$.

□

This result allows us to treat the system of exterior equations, the set of generators for the ideal, and the algebraic ideal as essentially equivalent objects. From here we may abuse notations and denote the system of equations as its corresponding generator and the generator set as its corresponding ideal.

2.3.3 Definition Let Σ_1 and Σ_2 be two sets of generators. If $I_{\Sigma_1} = I_{\Sigma_2}$, i.e., they generate the same ideal, we will say that the *generators* are *algebraically equivalents* .

We will use this definition to represent the system of exterior equations in a simplified way.

2.3.4 Definition Let Σ be a system of exterior equations and I_Σ the ideal which it generates. The *associated space* of the ideal I_Σ is defined by:

$$A\left(I_\Sigma\right) := \{v \in V \,|\, v \lrcorner \alpha \in I_\Sigma, \forall \alpha \in I_\Sigma\}$$

2.3.5 Definition The dual associated space, or *retracting space* of the ideal is defined by $C(I_\Sigma) = A\left(I_\Sigma\right)^\perp \subset V^*$.

Once the retracting space is determined one can find an algebraic equivalent system Σ' that is a subset of $\Lambda(C(I_\Sigma))$, the exterior algebra over the retracting space.

2.3.6 Theorem *Let $a_1, \ldots, a_n$ be a basis for V. Then the value of an alternating k-tensor $\omega \in \Lambda^k(V^*)$ is independent of a basis element a_i if and only if $a_i \lrcorner \omega \equiv 0$.*

Proof:
Let $\phi_1, \ldots, \phi_n$ be a dual basis to $a_1, \ldots, a_n$. Then ω can be written with respect to the dual basis as

$$\omega = \sum_J d_J \phi_{j_1} \wedge \phi_{j_2} \wedge \ldots \wedge \phi_{j_k} = \sum_J d_J \psi_J$$

where the sum is taken over all ascending k-tuples J. If a basis element ψ_J does not contain ϕ_i, then clearly $a_i \lrcorner \psi_J \equiv 0$.
If a basis element contains ϕ_i, then $a_i \lrcorner \wedge \phi_{j_1} \wedge \phi_{j_2} \wedge \ldots \wedge \phi_{j_k} \not\equiv 0$ because a_i can always be matched with ϕ_i through a permutation that affects only the sign. Consequently, $(a_i \lrcorner \omega) \equiv 0$ if and only if the coefficients d_J of all the terms containing ϕ_j are zero. ∎

2.3.7 Theorem (Characterization of retracting space) *Let Σ be a system of exterior equations and I_Σ its corresponding algebraic ideal. Then there exists an algebraically equivalent system Σ' such that $\Sigma' \subset \Lambda(C(I_\Sigma))$.*

Proof:
Let $v_1, \ldots, v_n$ be a basis for V and $\phi_1, \ldots, \phi_n$ be the dual basis, selected such that $v_{r+1}, \ldots, v_n$ span $A(I_\Sigma)$. Consequently $\phi_1, \ldots, \phi_r$ must span $C(I_\Sigma)$. By induction:
Consider α be any 1-tensor in I_Σ. With respect to the chosen basis, α can be written as

$$\alpha = \sum_{i=1}^{n} a_i \phi_i$$

Taking into account that $v \lrcorner \alpha \equiv 0 \mod I_\Sigma$ for all $v \in A\left(I_\Sigma\right)$, then $a_i = 0$ per $i = r+1, \ldots, n$. Hence,

$$\alpha = \sum_{i=1}^{r} a_i \phi_i$$

Therefore, all the 1-tensors in Σ are contained in $\Lambda^1(C(I_\Sigma))$.

Now suppose that all the tensors of degree less or equal than k in I_Σ are contained in $\Lambda(C(I_\Sigma))$. Let α be any $(k+1)$-tensor in I_Σ. We consider the tensor

$$\alpha' = \alpha - \phi_{r+1} \wedge (v_{r+1} \lrcorner \alpha)$$

The term $v_{r+1} \lrcorner \alpha$ is a k-tensor in I_Σ by the definition of associated space, and thus, by the induction hypothesis, it must be in $C(I_\Sigma)$. The wedge product of this term with ϕ_{r+1} belongs in $\Lambda(C(I_\Sigma))$. Furthermore:

$$v_{r+1} \lrcorner \alpha' = v_{r+1} \lrcorner \alpha - (v_{r+1} \lrcorner \phi_{r+1}) \wedge (v_{r+1} \lrcorner \alpha) + \phi_{r+1} \wedge (v_{r+1} \lrcorner (v_{r+1} \lrcorner \alpha)) \equiv 0$$

By the theorem (2.3.6), α' has no terms involving ϕ_{r+1}.

If we now replace α with α', the ideal generated will be unchanged since

$$\theta \wedge \alpha = \theta \wedge \alpha' + \theta \wedge \phi_{r+1} \wedge (v_{r+1} \lrcorner \alpha)$$

and $v_{r+1} \lrcorner \alpha \in I_\Sigma$.
We can continue this process for $v_{r+2}, \ldots, v_n$ to produce an $\hat{\alpha}$ that is a generator of I_Σ and is an element of $\Lambda(C(I_\Sigma))$. ■

2.3.8 Definition Given α a p-form, we define the *space of linear divisors* of α as:

$$L_\alpha = \{\omega \in V^* \,|\, \omega \wedge \alpha = 0\}$$

2.3.9 Theorem *Let I_Σ be an ideal generated by the set:*

$$\Sigma = \{\omega_1, \ldots, \omega_s, \Omega\}$$

where $\omega_i \in V^*$ and $\Omega \in \Lambda^2(V^*)$. Let r be the smallest integer such that

$$(\Omega)^{r+1} \wedge \omega_1 \wedge \ldots \wedge \omega_s = 0$$

Then the retracting space $C(I_\Sigma)$ has dimension $2r + s$.

Proof:
We consider the first case $s = 0$. Then,

$$\Sigma = \{\Omega\}, \quad \text{i } (\Omega)^{r+1} = 0$$

Since the ideal generated by Σ is defined as

$$I_\Sigma = \left\{ \pi \in \Lambda(V^*) \,|\, \pi = \sum_{i=1}^{n} \theta_i \wedge \Omega, \quad \theta_i \in \Lambda(V^*) \right\}$$

any element of I_Σ will be a linear combination of $\Omega, \Omega^2, \ldots, \Omega^r$.

Since $\Omega \in \Lambda(C(I_\Sigma))$ and $\Omega^r \in \Lambda^{2r}(C(I_\Sigma))$ then:

$$\dim(C(I_\Sigma)) \geq 2r$$

Let's consider $f : V \longrightarrow V^*$ a linear map defined as

$$f(x) = x \lrcorner \Omega, \quad x \in V$$

Note that the ideal generated by Σ does not contain any 1-form, hence,

$$x \lrcorner \Omega = 0 \Longleftrightarrow x \in A(I_\Sigma)$$

Which proves that

$$\ker\ f = A(I_\Sigma)$$

Therefore, dim(ker f) = $\dim(A(I_\Sigma))$. Since $A(I_\Sigma) = C(I_\Sigma)^\perp$, then:

$$\dim(\ker\ f) \leq n - 2r$$

On the other hand, for $s = 0$:

$$x \lrcorner \Omega^{r+1} = (r+1)(x \lrcorner \Omega) \wedge \Omega^r = 0$$

the last equality is true since $\Omega^{r+1} = 0$.

An element of the image of f belong in L_{Ω^r} since:

$$\text{im}\ f = \{\omega \in V^* \,|\, \omega = x \lrcorner \Omega, \quad x \in V\}$$

This condition implies that $\omega \wedge \Omega^p = (x \lrcorner \Omega) \wedge \Omega^r = 0$, then, $\omega \in L_{\Omega^r}$. Therefore, im $f \subset L_{\Omega^r}$.

Since Ω^r has degree $2r$ and has at most $2r$ linear divisors,

$$\dim(\text{im}\ f) \leq 2r$$

An elemental linear algebra result states that

$$\dim(\ker\ f) + \dim(\mathrm{im}\ f) = n$$

Hence, $\dim(\mathrm{im}\ f) = 2r$, $\dim(\ker\ f) = n - 2r$ and, consequently, $\dim(C(I_\Sigma)) = 2r$.

In the general case, we consider $W^* = \{\omega_1, \ldots, \omega_s\}$, that has dimension s.
Then $W = (W^*)^\perp \subset V$ and the quocient space V^*/W^* have a relation induced by the relation of V with V^*, and are dual vectorial spaces. By hypothesis:

$$\Omega^r \wedge \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_s \neq 0$$

and $\Omega^r \wedge \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_s \in \Lambda^{2r+s}(C(I_\Sigma))$, so that

$$\dim(C(I_\Sigma)) \geq 2r + s$$

The following linear map is considered

$$f' : W \xrightarrow{\ f\ } V^* \xrightarrow{\ \pi\ } V^*/W^*$$

where π is the projection to the quocient space and f is the map defined before.

As in the trivial case, we wish to find upper bounds for the dimensions of the kernel and image of f'.

Using the algebra result we know:

$$\dim(\ker\ f') + \dim(\mathrm{im}\ f') = \dim(W) = n - s$$

Reasoning similarly to the previous case, we find:

$$\begin{aligned} \dim(\ker\ f) &\leq n - 2r - s \\ \dim(\mathrm{im}\ f) &\leq 2r \end{aligned}$$

Consequently, $\dim(C(I_\Sigma)) = 2r + s$. ∎

2.4 Codistributions

2.4.1 Definition A smoothly *distribution* associates a subspace of the tangent space at each point $p \in M$. It is represented as the span of d smooth vector fields, as follows

$$\Delta = \langle f_1, \ldots, f_d \rangle$$

The *dimension* of the codistribution at a point is defined to be the dimension of the subspace $\Delta(p)$. A distribution is said to be *regular* if its dimension does not vary with p.

2.4.2 Definition A *codistribution* is defined as the map that associates at each point of the variety a set of 1-forms. This linear combination of 1-forms will be a subspace of the cotangent space T_p^*M. We denote the codistribution as

$$\Theta(p) = \langle \omega_1(p), \ldots, \omega_d(p) \rangle$$

There is notion of duality between distributions and codistributions which allow us to construct codistribution from distributions and vice versa.

Given a distribution Δ, for each p in a neighborhood U, consider all the 1-forms which pointwise annihilate all vectors in $\Delta(p)$,

$$\Delta^\perp(p) = \langle \omega(p) \in T_p^*M : \omega(p)(f) = 0, \quad \forall f \in \Delta(p) \rangle$$

Clearly, $\Delta^\perp(p)$ is a subspace of T_p^*M and is therefore a codistribution. We call $\Delta^\perp$ the annihilator or dual of Δ. Conversely, given a codistribution Θ, we construct the annihilating or dual distribution pointwise as

$$\Theta^\perp(p) = \langle v \in T_pM : \omega(p)(v) = 0, \quad \forall \omega(p) \in \Omega(p) \rangle$$

3 Exterior differential systems

3.1 Exterior algebra on a manifold

The space of all forms on a manifold M,

$$\Omega(M) = \Omega^0(M) \oplus \cdots \oplus \Omega^n(M),$$

together with the wedge product is called *exterior algebra in M*. An algebraic ideal of this algebra is defined as a subspace I such that if $\alpha \in I$ then $\alpha \wedge \beta \in I$ for any $\beta \in \Omega(M)$.

3.1.1 Definition An ideal $I \subset \Omega(M)$ is said to be *closed* with respect to exterior differentiation if and only if

$$\alpha \in I \Rightarrow d\alpha \in I,$$

or more compactly, if $dI \subset I$. An algebraic ideal which is closed with respect to exterior differentiation is called a *differential ideal*.

A finite collection of forms, $\Sigma = \{\alpha_1, \ldots \alpha_K\}$ generates an algebraic ideal

$$I_\Sigma = \left\{ \omega \in \Omega(M) \,|\, \omega = \sum_{i=1}^{K} \theta_i \wedge \alpha_i \text{ for some } \theta_i \in \Omega(M) \right\}.$$

We also can talk about the differential ideal generated by Σ. Thus, if S_d denote the collection of all differential ideals containing Σ is defined to be the smallest differential ideal containing Σ:

$$\mathcal{I}_\Sigma = \bigcap_{I \in S_d} I.$$

3.1.2 Theorem *Let Σ be a finite collection of forms and let $\mathcal{I}_\Sigma$ the differential ideal generated by Σ. Define the collection*

$$\Sigma' = \Sigma \cup d\Sigma$$

and denote the algebraic ideal which it generates by $I_{\Sigma'}$.

Proof:

By definition $\mathcal{I}_\Sigma$ is closed with respect to exterior differentiation, so $\Sigma' \subset \mathcal{I}_\Sigma$.

Consequently, $I_{\Sigma'} \subset \mathcal{I}_\Sigma$. The ideal $I_{\Sigma'}$ is a closed with respect to exterior differentiation and contains Σ by construction. Therefore, from the definition of $\mathcal{I}_\Sigma$ we have that $\mathcal{I}_\Sigma \subset I_{\Sigma'}$. ■

The associated space and retracting space of an ideal in $\mathcal{I}_\Sigma$ is called *characteristic distribution of Cauchy* and is denoted by $\mathcal{A}(\mathcal{I}_\Sigma)$.

3.2 Exterior Differential Systems

In the previous section we have introduced systems of exterior equations on a vector space V and characterized their solutions as subspaces of V. We are now ready to define a similar notion for a collection of differential forms defined on a manifold M. The basic problem will be to study the integral submanifolds of M which satisfy the constraints represented by the exterior differential system.

3.2.1 Definition An *exterior differential system* is a finite collection of equations

$$\alpha_1 = 0, \dots, \alpha_r = 0,$$

where each $\alpha_i \in \Omega^k(M)$ is a smooth $k-$form. A *solution to an exterior differential system* is any submanifold N o f M which satisfies $\alpha_i(x)|_{T_xN} \equiv 0$ for all $x \in N$ and all $i \in \{1, \dots, r\}$.

An exterior differential system can be viewed pointwise as a system of exterior equations on T_pM. In view of this, one might expect that a solution would be defined as a distribution on the manifold. The trouble with this approach is that most distributions are not integrable, and we want our solution set to be a collection of integral submanifolds. Therefore, we will restrict our solution set to integrable distributions.

3.2.2 Theorem *Given an exterior differential system*

$$\alpha_1 = 0, \dots, \alpha_K = 0$$

and the corresponding differential ideal $\mathcal{I}_\Sigma$ generated by the collection of forms

$$\Sigma = \{\alpha_1, \dots, \alpha_K\},$$

an integral submanifold N of M solves the system of exterior equations if and only if it also solves the equation $\pi = 0$ for each $\pi \in \mathcal{I}_\Sigma$.

Proof:
If an integral submanifold N of M is a solution to Σ, then for all $x \in N$ and all $i \in \{1, \dots, K\}$,

$$\alpha_i(x)|_{T_xN} \equiv 0.$$

Taking the exterior derivative we get

$$d\alpha_i(x)|_{T_xN} \equiv 0.$$

Hence, the submanifols also satisfies the exterior differential system

$$\alpha_1 = 0, \ldots, \alpha_K = 0, \quad d\alpha_1 = 0, \ldots, d\alpha_K = 0.$$

By the theorem 3.1.2 we know that the differential ideal generated by Σ is equal to the algebraic ideal generated by the above system. Therefore, the theorem 2.3.2 tells us that every solution N to Σ is also a solution for every element of $\mathcal{I}_\Sigma$.Conversely, if N solves the equation $\pi = 0$ for every $\pi \in \mathcal{I}_\Sigma$ then in particular it must solve Σ. ■

This theorem allows us to work either with the generators of an ideal or with the ideal itself. In fact some authors define exterior differential systems as differential ideals of $\Omega(M)$. Because a set of generators Σ generates both a differential ideal $\mathcal{I}_\Sigma$ and a algebraic ideal I_Σ, we can define two different notions of equivalence for exterior differential systems. Two exterior differential systems Σ_1 and Σ_2 are said to be *algebraically equivalent* if they generate the same algebraic ideal. i.e, $\mathcal{I}_{\Sigma_1} = \mathcal{I}_{\Sigma_2}$. Two exterior differential systems Σ_1 and Σ_2 are said to be *equivalent* if they generate the same algebraic ideal. i.e, $\mathcal{I}_{\Sigma_1} = \mathcal{I}_{\Sigma_2}$. Intuitively, we want to think of two exterior differential systems as equivalent if they have the same solution set. Therefore, we will usually discuss equivalence in the latter sense.

3.3 Pfaffian systems

Pfaffian systems are of particular interest because they can be used to represent a set of first-order ordinary differential equations.

3.3.1 Definition An exterior differential system of the form

$$\alpha_1 = \alpha_2 = \cdots = \alpha_s = 0,$$

where the α_i are independent 1-forms on an n-dimensional manifold M, is called a *Pfaffian system* of codimension $n-s$. If $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $\Omega^1(M)$, then the set $\{\alpha_{s+1}, \ldots, \alpha_n\}$ is called a *complement* to the Pfaffian system

An *independence condition* is a 1-form τ that is required to be nonzero along integral curves of the Pfaffian system. That is $\alpha_i(c(t))(c'(t)) = 0$, then $\tau(c(t))(c'(t)) \neq 0$. The 1-forms $\alpha_1, \ldots, \alpha_s$, generate the algebraic ideal

$$I = \{\sigma \in \Omega(M) : \sigma \wedge \alpha_1 \wedge \ldots \wedge \alpha_s = 0\}.$$

The algebraic ideal generated by the 1-forms α_i is also a differential ideal if the following conditions are satisfied.

3.3.2 Definition A set of linearly independent 1-forms $\alpha_1, \ldots, \alpha_s$ in a neighborhood of a point is said to satisfy the Frobenius condition if one of the following equivalent conditions holds:

1) $d\alpha_i$ is a linear combination of $\alpha_1, \ldots, \alpha_s$.

2) $d\alpha_i \wedge \alpha_1 \wedge \ldots \wedge \alpha_s = 0$ for $1 \leq i \leq s$.

3) $d\alpha_i = \sum_{j=1}^s \theta_j \wedge \alpha_j$.

When $d\alpha_i$ is a linear combination of $\alpha_1, \ldots, \alpha_s$, the following expression is frequently used

$$d\alpha_i \equiv 0 \mod \alpha_1, \ldots, \alpha_s \quad 1 \leq i \leq s$$

where the *mod* operation is implicitly performed over the algebraic ideal generated by α_i.

3.3.3 Theorem (Frobenius Theorem for Codistributions) *Let I be an algebraic ideal generated by the independent 1-forms $\alpha_1, \ldots, \alpha_{n-r}$ which satisfies the Frobenius condition. Then in a neighborhood of x there exist functions $y_1, \ldots, y_n$ such that*

$$I = \{\alpha_1, \ldots, \alpha_{n-r}\} = \{dy_{r+1}, \ldots, dy_n\}.$$

Proof:
We will prove the therem by induction on r. Let $r = 1$. Then the subspace $W_x^\perp \subset T_x, \quad x \in M$, is of dimension 1. The equations of the differential system is written in the classical form

$$\frac{dx_1}{X_1(x)} = \cdots = \frac{dx_n}{X_n(x)},$$

where the functions $X_i(x_1, \ldots, x_n)$, not all zero, are the coefficients of a vector field $X = \sum_i X_i(x)\partial/\partial x_i$ spanning $W_x^\perp$. By the *flow box coordinate theorem*, we can choose cooridinates $y_1, \ldots, y_n$, such that $W_x^\perp$ is spanned by the vector $\partial/\partial y_1$, then W_x is spanned by $dy_2, \ldots, dy_n$. The latter clearly form a set of generators if I. Notice that in this case the Frobenius condition is void.

Suppose $r \geq 2$ and the theorem be true for $r - 1$. Let $x_i, 1 \leq i \leq n$, be local coordinates such that

$$\alpha_1, \ldots, \alpha_{n-r}, dx_r$$

are linearly independent. The differential system defined by these $n - r + 1$ forms also satisfies the Frobenius condition. By the induction hypothesis there are coordinates y_i so that

$$dy_r, dy_{r+1}, \ldots, dy_n$$

are a set of generators of the corresponding differential ideal. It follows that dx_r is a linear combination of these forms or that x_r is a function of $y_r, \ldots, y_n$. Without loss of generality we suppose

$$\partial x_r/\partial y_r \neq 0.$$

Since

$$dx_r = \frac{\partial x_r}{\partial y_r} + \sum_i \frac{\partial x_r}{\partial y_{r+1}} dy_{r+1}, \quad 1 \leq i \leq n - r$$

we may now solve dy_r in terms of dx_r and $dy_{r+1}, \ldots, dy_n$. Since $\alpha_1, \ldots, \alpha_{n-r}$ are linear combinations of $dy_r, \ldots, dy_n$, the α_i can be expressed in the form

$$\alpha_i = \sum_j a_{ij} dy_{r+j} + b_i dx_r, \quad 1 \leq i, j \leq n - r.$$

Since α_i and dx_r are linearly independent, the matrix (a_{ij}) must be non-singular. Hence we can find a new set of generators for I in the form I

$$\alpha_i' = dy_{r+i} + p_i dx_r, \quad 1 \le i \le n - r,$$

and the Frobenius condition remains satisfied. Exterior differentiation gives

$$d\alpha_i' = dp_i \wedge dx_r \equiv \sum_{1 \le \lambda \le r-1} \frac{\partial p_i}{\partial y_\lambda} du_\lambda \wedge dx_r \equiv 0, \qquad \mathrm{mod}\ \alpha_1', \ldots \alpha_{n-r}'.$$

It follows that

$$\partial p_i / \partial y_\lambda = 0, \quad 1 \le i \le n - r, \quad 1 \le \lambda \le r - 1,$$

which means that p_i are functions of $y_r, \ldots, y_n$. Hence in the y-coordinates we are studying a system of $n - r$ 1-forms involving only the $n - r + 1$ coordinates $y_r, \ldots, y_n$. This reduces to the situation settled at the beggining of this proof. Hence the induction is complete. ■

The following corollary is a version of the classical Frobenius theorem that will be used in section 6.

3.3.4 Corollary *Let $y_1, \cdots y_m$ be functions whose differentials are linearly independent from linearly independent 1-forms $\alpha_1, \ldots, \alpha_p$ and satisfying the relative Frobenius conditions*

$$d\alpha_u \wedge \alpha_1 \wedge \cdots \alpha_p \wedge dy_1 \wedge \cdots \wedge dy_m = 0 \quad i \le i \le m$$

Then setting

$$\alpha = (\alpha_1, \cdots, \alpha_p)^t, \quad Y = (y^1, \cdots y^m)^t$$

there exists a vector of functions $Z = (z_1, \cdots, z_p)^t$ a $p \times p$ matrix A and a $p \times m$ matrix B, such that

$$\alpha = A\, dZ + B\, dY$$

For more general exterior differential systems we have the following integrability results.

3.3.5 Proposition *Is the Cauchy characteristic distribution $\mathcal{A}(\mathcal{I}_\Sigma)$ of $\mathcal{I}_\Sigma$ has constant dimension r in a neighborhood x, then the distribution $\mathcal{A}(\mathcal{I}_\Sigma)$ is integrable.*

3.3.6 Theorem *Let $\mathcal{I}$ be a differential ideal whose retracting space $\mathcal{C}(\mathcal{I})$ has a constant dimension $s = n - r$. There is a neighborhood in which there are coordinates $(x_1, \ldots, x_r; y_1, \ldots, y_n)$ such that $\mathcal{I}$ has a set of generators which are forms in $y_1, \ldots, y_s$ and their differentials.*

Proof:
By proposition 3.3.5 the differential system defined by $\mathcal{C}(\mathcal{I})$, or what is the same, the distribution defined by $\mathcal{A}(\mathcal{I})$,is completely integrable. We may choose coordinates $(x_1, \ldots, x_r; y_1, \ldots, y_s)$ so that the foliation si defined is given by

$$y_\sigma = \text{const}, \quad 1 \le \sigma \le s.$$

By the retraction theorem, $\mathcal{I}$ has a set of generators which are forms in $dy_\sigma, 1 \le \sigma \le s$. But their coefficients may involve $x_\rho, \quad 1 \le \rho \le r$. The theorem follows when we show that we can choose a new set of generators for $\mathcal{I}$ which are forms in the y_σ coordinates in which the x_ρ do not enter. To exclude the trivial case we suppose the $\mathcal{I}$ is a proper ideal, so that it contains no non-zero functions.

Let $\mathcal{I}_q$ be the set of q-forms in $\mathcal{I}$, $q = 1, 2, \ldots$. Let $\varphi_1, \ldots, \varphi_p$ be the linearly independent 1-forms in $\mathcal{I}_1$ such that any form in $\mathcal{I}_1$ is a linear combination. Since $\mathcal{I}$ is closed, $d\varphi_i \in \mathcal{I}$, $1 \le i \le p$. For a fixed ρ we have that $\frac{\partial}{\partial x_\rho} \in \mathcal{A}(\mathcal{I})$, which implies

$$\frac{\partial}{\partial x_\rho} \lrcorner\, d\varphi_i = L_{\partial/\partial x_\rho} \varphi_i \in \mathcal{I}_1,$$

since the left-hand side is of degree 1. It follows that

$$\frac{\partial \varphi_i}{\partial x_\rho} = L_{\partial/\partial x_\rho} \varphi_i = \sum_j a_{ij} \varphi_j, \quad 1 \le i, j \le p \tag{3.1}$$

where the left hand side stands for the form obtained from φ_i by taking the partial derivatives of the coefficients with respect to x_ρ.

For this fixed ρ we regard x_ρ as the variable $x_1, \ldots, x_{\rho-1}, x_{\rho+1}, \ldots, x_r, y_1, \ldots, y_s$ as parameters. Consider the system of ordinary differential equations

$$\frac{dz_i}{dx_\rho} = \sum_j a_{ij} z_j, \quad 1 \le i, j \le p. \tag{3.2}$$

Let $z_i^{(k)}, 1 \le k \le p$, be a fundamental system of solutions, so that

$$\det \left(z_i^{(k)} \right) \neq 0.$$

We shall replace φ_i by the $\widetilde{\varphi}_k$ defined by

$$\varphi_i = \sum z_i^{(k)} \widetilde{\varphi}_k. \tag{3.3}$$

By differentiating (3.3) with respect to x_ρ and using (3.1) and (3.2) we get

$$\frac{\partial \widetilde{\varphi}_k}{\partial x_\rho} = 0,$$

so that $\widetilde{\varphi}_k$ does not involve x_ρ. Applying the same process to the other x, we arrive at a set of generators $\mathcal{I}_1$ which are forms in y_σ.

Sppose this process carried out for $\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}$, so that they consist of forms in y_σ. Let $\mathcal{J}_{q-1}$ the ideal generated by per $\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}$. Let $\psi_\alpha \in \mathcal{I}_q$, $1 \le \alpha \le r$, linearly independents $\mod \mathcal{J}_{q-1}$, such that any q-form of $\mathcal{I}_q$ is congruent $\mod \mathcal{J}_{q-1}$to a linear combination of them. By the above argument such forms include

$$\frac{\partial}{\partial x_\rho} \lrcorner\, d\psi_\alpha = L_{\partial/\partial x_\rho} \psi_\alpha.$$

Hence we have

$$\frac{\partial \psi_\alpha}{\partial x_\rho} \equiv \sum b_\alpha^\beta \psi_\beta, \quad \mod \mathcal{J}_{q-1}, \quad 1 \le \alpha, \beta \le r.$$

By using the above argument, we can replace the ψ_α by $\widetilde{\psi}_\beta$ such that

$$\frac{\partial \widetilde{\psi}_\alpha}{\partial x_\rho} \in \mathcal{J}_{q-1}.$$

This means that we can write

$$\frac{\partial \widetilde{\psi}_\alpha}{\partial x_\rho} = \sum_h \eta_\alpha^h \wedge \omega_\alpha^h,$$

where $\eta_\alpha^h \in \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{q-1}$ and are therefore forms in y_σ. Let θ_α^h definied by

$$\frac{\partial \theta_\alpha^h}{\partial x_\rho} = \omega_\alpha^h.$$

Then the forms

$$\widetilde{\widetilde{\psi}}_\alpha = \widetilde{\psi}_\alpha - \sum_h \eta_\alpha^h \wedge \theta_\alpha^h$$

do not involve x_ρ, and can be used to replace ψ_α. Applying this process to all $x_\rho, 1 \leq \rho \leq r$, we find a set of generators for $\mathcal{I}_q$, which are forms only in y_σ. ■

3.4 Derived flags

If the algebraic ideal generated by a Pfaffian system does not satisfy the Frobenius condition, then it is not a differential ideal. However, there may exist a differential ideal which is a subset of the algebraic ideal. This subideal can be found by taking the derived flag of the Pfaffian system. Let $I^{(0)} = \{\omega_1, \ldots, \omega_s\}$ be the algebraic ideal generated by independent 1-forms $\omega_1, \ldots \omega_s$. We define $I^{(1)}$ as

$$I^{(1)} = \{\lambda \in I^{(0)} : d\lambda \equiv 0 \mod I^{(0)}\} \subset I^{(0)}.$$

The ideal $I^{(1)}$ is called the *first derived system.* The analogue of the first derived system from the distribution point of view is given by the following theorem.

3.4.1 Theorem *If $I^{(0)} = \Delta^\perp$, then $I^{(1)} = (\Delta + [\Delta, \Delta])^\perp$.*

One may inductively continue this procedure of obtaining derived systems and define

$$I^{(2)} = \{\lambda \in I^{(1)} : d\lambda \equiv 0 \mod I^{(1)}\} \subset I^{(1)}$$

or, in general

$$I^{(k+1)} = \{\lambda \in I^{(k)} : d\lambda \equiv 0 \mod I^{(k)}\} \subset I^{(k)}.$$

This procedure results in a nested sequence of codistributions

$$I^{(k)} \subset I^{(k-1)} \subset \cdots \subset I^{(1)} \subset I^{(0)}. \tag{3.4}$$

We can also generalize theorem 3.4.1. If we define

$$\begin{aligned}\Delta_0 &= (I^{(0)})^{\perp}\\ \Delta_1 &= (I^{(1)})^{\perp}\\ &\vdots\\ \Delta_k &= (I^{(k)})^{\perp}\end{aligned}$$

then it is not hard to show that if $I^{(k)} = \Delta_k^{\perp}$ then $I^{(k+1)} = (\Delta_k + [\Delta_k, \Delta_k])^{\perp}$. The proof of this fact is similar to the proof of theorem 3.4.1 but uses a more general form of Cartan's formula. The sequence of decreasing codistributions (3.4) , called the *derived flag* of $I^{(0)}$, is associated with a sequence of increasing distributions, called the *filtration* of Δ_0 (or the *coderived coflag* of $I^{(0)}$),

$$\Delta_k \supset \Delta_{k-1} \supset \cdots \supset \Delta_1 \supset \Delta_0.$$

3.4.2 Definition Let I be an algebraic ideal corresponding to a Pfaffian system. We define the *derived length* of I as the smallest integer N such that

$$I^{(N)} = I^{(N+1)}$$

A basis for a codistribution I is simply a set of generators for I. A basis of 1-forms α_j for I is said to be *adapted to the derived flag* if a basis for each derived system $I^{(j)}$ can be chosen to be some subset of the α_j. The codistribution $I^{(N)}$ is always integrable by definition since

$$dI^{(N)} \equiv 0 \mod I^{(N)}.$$

The codistribution $I^{(N)}$ is the largest integrable subsystem in I.Therefore, if

$I^{(N)} \neq \{0\}$ then there exist function $h_1, \ldots, h_r$ such that $\{dh_1, \ldots, dh_r\} \subset I$. As a result, if a Pfaffian system contains an integrable subsystem $I^{(N)} \neq \{0\}$, which is spanned by the 1-forms $dh_1, \ldots, dh_r$, then the integral curves of the system are constrained to satisfy the following equations for some constants k_i,

$$dh_i = 0 \Longrightarrow h_i = k_i, \quad \text{per a} \quad 1 \leq i \leq r,$$

or equivalently, trajectories of the system must lie on the manifold,

$$M = \{x : h_i(x) = k_i \quad \text{per a} \quad 1 \leq i \leq r\}.$$

In particular, this implies that if $I^{(N)} \neq 0$, it is not possible to find an integral curve of the Pfaffian system which connects a configuration $x(0) = x_0$ to another configuration $x(1) = x_1$ unless the initial and final configurations satisfy

$$h_i(x_0) = h_i(x_1) \quad \text{per a} \quad 1 \leq i \leq r.$$

4
The Goursat normal forms

Now that we have defined an exterior differential system and introduced some tools for analyzing them, we are ready to study some important normal forms for exterior differential systems. We will restrict ourselves to Pfaffian systems. The first normal form which we introduce, the Pfaffian form, is restricted to systems of only one equation. The Engel form applies to two equations on a four-dimensional space, and the Goursat form is for $n-2$ equations on an n-dimensional space. The extended Goursat normal form is defined for systems with codimension greater than two. The Goursat normal forms can be thought of as the generalization of linear systems. Their study will lead us to the study of linearization of control.

4.1 Systems of one equation

We will first study Pfaffian systems of codimension $n-1$, or systems consisting of a single equation

$$\alpha = 0$$

where α t is a 1-form on a manifold M. In some chart (U, x) of a point $p \in M$ the equation can be expressed as

$$a_1(x)dx_1 + a_2(x)dx_2 + \cdots + a_n(x)dx_n = 0.$$

In order to understand the integral manifolds of this equation we will attempt to express α in a normal form by performing a coordinate transformation.

4.1.1 Definition Let $\alpha \in \Omega^1(M)$. The integer r defined as

$$(d\alpha)^r \wedge \alpha \neq 0$$

$$(d\alpha)^{r+1} \wedge \alpha = 0$$

is called *rank* of α.

The following theorem allow us, under a rank condition, to write α in a normal form.

4.1.2 Theorem (Pfaff theorem) *Let $\alpha \in \Omega^1(M)$ have a constant rank r in a neighborhood of p. Then there exists a coordinate chart (U, z) such that in these coordinates $\alpha = dz_1 + z_2dz_3 + \cdots + z_{2r}dz_{2r+1}$.*

Proof:
Let $\mathcal{I}$ be the differential ideal generated by α. From theorem 2.3.9 the retracting space of $\mathcal{I}$ has dimension $2r+1$. By the theorem 3.3.6 there exist local coordinates $y_1, \ldots, y_n$ such that $\mathcal{I}$ has a set of generators in $y_1, \ldots, y_{2r+1}$. Then, by dimension count, any function f_1 of those $2r+1$ coordinates results in

$$(d\alpha)^r \wedge \alpha \wedge df_1 = 0.$$

Now, let $\mathcal{I}_1$ be the ideal generated by $\{df_1, \alpha, d\alpha\}$. If $r = 0$, then the result follows from the Frobenius theorem 3.3.3. If $r > 0$, then the forms df_1 and α a must be linearly independent, since α is not integrable. Applying theorem 2.3.9 to $\mathcal{I}_1$, let r_1 be the smallest integer such that

$$(d\alpha)^{r_1+1} \wedge \alpha \wedge df_1 = 0.$$

Clearly, $r_1 + 1 \leq r$. Furthermore, the equality sign must hold because $(d\alpha)^r \wedge \alpha \neq 0$. Applying theorem 3.3.6 to $\mathcal{I}_1$ there exists a function f_2 such that

$$(d\alpha)^{r-1} \wedge \alpha \wedge df_1 \wedge df_2 = 0.$$

Repeating this process, we find r functions $f_1, f_2, \ldots, f_r$ satisfying

$$\begin{aligned} d\alpha \wedge \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r &= 0, \\ \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r &\neq 0. \end{aligned}$$

Finally, let I be the ideal $\{df, \ldots, df_r, \alpha, d\alpha\}$. Its retracting space $\mathcal{C}(I_r)$ is of dimension $r+1$. There is a function f_{r+1} such that:

$$\begin{aligned} \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r \wedge df_{r+1} &= 0, \\ df_1 \wedge df_2 \wedge \cdots \wedge df_r \wedge df_{r+1} &\neq 0. \end{aligned}$$

By modifying α by a factor, we can write

$$\alpha = df_{r+1} + g_1 df_1 + \cdots + g_r df_r.$$

Because $(d\alpha)^r \wedge \alpha \neq 0$, the functions $f_1, \ldots, f_{r+1}, g_1, \ldots, g_r$ are independent. The result then follows by setting

$$z_1 = f_{r+1} \qquad z_{2i} = g_i \qquad z_{2i+1} = kf_i$$

for $1 \leq i \leq r$. ■

The following theorem is similar to Pfaff's theorem and simply expresses the result in a more symmetric form.

4.1.3 Theorem (Symmetric Version of Pfaff Theorem) *Given any $\alpha \in \Omega^1(M)$ with constant rank r in a neighborhood U of p, then there exists coordinates $z, y_1, \ldots, y_r, x_1, \ldots, x_r$ such that*

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y_i dx_i - x_i dy_i).$$

The Pfaffian system $\alpha = 0$ in a manifold M is said to have the local accessibility property if every point $x \in M$ thas a neighborhood U such that every point in U can be joined to x by an integral curve. The following theorem answers the question of when this Pfaffian system has the local accessibility property.

4.1.4 Theorem (Caratheodory Theorem) *The Pfaffian system*

$$\alpha = 0,$$

on α where a has constant rank, has the local accessibility property if and only if

$$\alpha \wedge d\alpha \neq 0.$$

Proof:
The above condition simply says that the rank of α must be greater than or equal to 1. If α has rank 0 then $d\alpha \wedge \alpha = 0$, and therefore by the Frobenius theorem (theorem 3.3.3), we can write

$$\alpha = dh = 0$$

for some function h. The integral curves are of the form $h = c$ for any arbitrary constant c.. Since we can only join points $p, q \in M$ for which $h(p) = h(q)$, we do not have the local accessibility property.
Conversely, let α have rank $r \geq 1$. From theorem 4.1.3, we can find coordinates $z, x_1, \ldots, x_r, y_1, \ldots, y_r, u_1, \ldots, u_s$ in some neighborhood U, with $2r + s + 1$ as dimension of M, such that

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y_i dx_i - x_i dy_i) = 0,$$

and therefore

$$dz = \frac{1}{2}\sum_{i=1}^{r}(y_i dx_i - x_i dy_i).$$

Given any two points $p, q \in U$ we must find integral curve $c : [0,1] \longrightarrow U$ with $c(0) = p$ i $c(1) = q$. Since we are working locally, we can assume that the initial point p is the origin: $z(p) = x_i(p) = y_i(p) = u_i(p) = 0$. Let the final point q be defined by $z(q) = z_1, x_i(q) = x_{1i}, y_i(q) = y_{1i}, u_i(q) = u_{1i}$. Because the expression of the one-form a does not depend on the u^i coordinates, we can choose the curve tu_{1i} to connect thes u_i coordinates of p and q.

In the (x_i, y_i) plane there are many curves$(x_i(t), y_i(t))$ that join the origin with the desired point (x_{1i}, y_{1i}). We need to find one which steers the z coordinate to z_1. In order to satisfy the equation $\alpha = 0$, we must have that

$$dz = \frac{1}{2}\sum_{i=1}^{r}\left(x_i dy_i - y_i dx_i\right).$$

Integrating this equation gives

$$z(t) = \frac{1}{2}\int_0^t \sum_{i=1}^{r}\left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt}\right) dt = \frac{1}{2}\sum_{i=1}^{r} A_i,$$

where A_i is the area enclosed by the curve $(x_i(t), y_i(t))$ and the chord joining the origin with (x_{1i}, y_{1i}). To reach the point q, the curve $(x_i(t), y_i(t))$ must satisfy $z(1) = z_1$. Geometrically, it is clear that a curve

$(x_i(t), y_i(t))$ linking the points p and q while enclosing the area prescribed by z_1 will always exist. Thus, the integral curve $c(t)$ given by

$$(z(t), x_1(t), \ldots, x_r(t), y_1(t), \ldots, y_r(t), tu_1(t), \ldots, tu_s(t))$$

has $c(0) = p$ i $c(1) = q$ and satisfies the equation $\alpha = 0$, and the system therefore has the local accessibility property. ■

4.2 Systems of codimension two

We now consider Pfaffian systems of codimension two. We are again interested in performing coordinate changes so that the generators of these Pfaffian systems are in some normal form.

4.2.1 Theorem (Engels theorem) *Let I be a dimension two codistribution, spanned by*

$$I = <\alpha_1, \alpha_2>$$

of four variables. If the derived flag satisfies

$$\dim I^{(1)} = 1,$$

$$\dim I^{(2)} = 0,$$

then there exist coordinate z_1, z_2, z_3, z_4 such that

$$I = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}.$$

Proof:
Choose a basis for I that is adapted to the derived flag; that is $I^{(0)} = I = \{\alpha_1, \alpha_2\}$, $I^{(1)} = \{\alpha_1\}$ and $I^{(2)} = \{0\}$. Choose α_3 and α_4 to complete the basis. Since $I^{(2)} = \{0\}$ we have

$$d\alpha_1 \wedge \alpha_1 \neq 0,$$

while

$$(d\alpha_1)^2 \wedge \alpha_1 = 0,$$

since it is a 5-form on a 4-dimensional space. Therefore, α_1 has rank 1. By Pfaff's theorem, we know that there exists a coordinate change such that

$$\alpha_1 = dz_4 - z_3 dz_1.$$

Taking the exterior derivative, we have that

$$d\alpha_1 = -dz_3 \wedge dz_1 = dz_1 \wedge dz_3.$$

Now, since $\alpha_1 \in I^{(1)}$, he definition of the first derived system will imply that

$$d\alpha_1 \wedge \alpha_1 \wedge \alpha_2 = 0,$$

and thus

$$dz_1 \wedge dz_3 \wedge \alpha_1 \wedge \alpha_2 = 0.$$

Therefore, α_2 must be a linear combination of dz_1, dz_3 and α_1:

$$\alpha_2 \equiv a(x)dz_3 + b(x)dz_1 \mod \alpha_1.$$

By definition, this means that

$$\alpha_2 + \lambda(x)\alpha_1 = a(x)dz_3 + b(x)dz_1.$$

Now if either $a(x) = 0$ or $b(x) = 0$ will impliy that $d\alpha_2 \wedge \alpha_1 \wedge \alpha_2 = 0$ and thus the flag assumptions are violated because if $I^{(0)} = \{\alpha_1, \alpha_2\}$ and $I^{(1)} = \{\alpha_1\}$ implies $d\alpha_2 \not\equiv 0 \mod \alpha_1, \alpha_2$. Thus $a(x) \neq 0$, Then

$$\frac{1}{a(x)}\alpha_2 + \frac{\lambda(x)}{a(x)}\alpha_1 = dz_3 + \frac{b(x)}{a(x)}dz_1,$$

and if we set $z_2 = -\frac{b(x)}{a(x)}$ then

$$\frac{1}{a(x)}\alpha_2 + \frac{\lambda(x)}{a(x)}\alpha_1 = dz_3 - z_2 dz_1,$$

and thus

$$I = \{\alpha_1, \alpha_2\} = \left\{\alpha_1, \frac{1}{a(x)}\alpha_2 + \frac{\lambda(x)}{a(x)}\alpha_1\right\} = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}.$$

■

It should be noted that the only place the dimension assumption is used in the proof is to guarantee that $(d\alpha_1)^2 \wedge \alpha_1 = 0$. If α_1 as rank 1, this equality holds by definition.

4.2.2 Corollary *Let $I = \{\alpha_1, \alpha_2\}$ be a two-dimensional codistribution. If the derived flag satisfies* $\dim I^{(1)} = 1$ *and* $\dim I^{(2)} = 0$ *and $\alpha_1 \in I^{(1)}$ has rank 1, then there exist coordinates z_1, z_2, z_3, z_4 such that*

$$I = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}.$$

Proof: The proof is deduced from the Engel's theorem.

Engel's theorem can be generalized to a system with n configuration variables and $n - 2$ constraints.

4.2.3 Theorem (Goursat Normal Form) *Let I be a Pfaffian system spanned by s 1-forms*

$$I = \{\alpha_1, \dots, \alpha_s\},$$

on a space of dimension $n = s + 2$ Suppose that there exists an integrable form π with $\pi \neq 0 \mod I$ satisfying the Goursat congruences,

$$\begin{aligned} d\alpha_i &\equiv -\alpha_{i+1} \wedge \pi \mod \alpha_1, \dots, \alpha_i, \quad 1 \leq i \leq s-1, && (4.1) \\ d\alpha_s &\not\equiv 0 \mod I. && (4.2) \end{aligned}$$

Then there exists a coordinate system $z_1, z_2, \dots, z_n$ in which the Pfaffian system is in Goursat normal form:

$$I = \{dz_3 - z_2 dz_1, dz_4 - z_3 dz_1, \dots, dz_n - z_{n-1} dz_1\}.$$

Proof:
The Goursat congruences can be expressed a

$$\begin{aligned}
d\alpha_1 &\equiv -\alpha_2 \wedge \pi \mod \alpha_1,\\
d\alpha_2 &\equiv -\alpha_3 \wedge \pi \mod \alpha_1, \alpha_2,\\
&\vdots\\
d\alpha_{s-1} &\equiv -\alpha_s \wedge \pi \mod \alpha_1, \alpha_2, \ldots, \alpha_{s-1},\\
d\alpha_s &\equiv -\alpha_{s+1} \wedge \pi \mod \alpha_1, \alpha_2, \ldots, \alpha_s,
\end{aligned}$$

wher $\alpha_{s+1} \notin I$. It can be shown that $\{\alpha_{s+1}, \pi\}$ must form a complement to I. This basis satisfies the Goursat congruences and is adapted to the derived flag of I:

$$\begin{aligned}
I^{(0)} &= \{\alpha_1, \alpha_2, \ldots, \alpha_s\},\\
I^{(1)} &= \{\alpha_1, \ldots, \alpha_{s-1}\},\\
&\vdots\\
I^{(s-1)} &= \{\alpha_1\},\\
I^{(s)} &= \{0\}.
\end{aligned}$$

From the Goursat congruences,

$$d\alpha_1 \equiv -\alpha_2 \wedge \pi \mod \alpha_1,$$

which means that

$$d\alpha_1 = -\alpha_2 \wedge \pi + \alpha_1 \wedge \eta$$

for some one-form η. But then we have that

$$\begin{aligned}
d\alpha_1 \wedge \alpha_1 &= -\alpha^2 \wedge \pi \wedge \alpha_1 \neq 0,\\
(d\alpha_1)^2 \wedge \alpha_1 &= 0
\end{aligned}$$

which means that α_1 has rank 1. We can therefore apply Pfaff's theorem and suppose that multiplying α_1 by a certain factor if it is necessary, α_1 can be expressed as

$$\alpha_1 = dz_n - z_{n-1} dz_1$$

or some choice of z_1, z_{n-1}, z_n. Furthermore, by Corollary 4.2.2 we can express α_2 as

$$\alpha_2 = dz_{n-1} - z_{n-2} dz_1. \tag{4.3}$$

In these new coordinates we have

$$d\alpha_1 \wedge \alpha_1 = -dz_{n-1} \wedge dz_1 \wedge dz_n.$$

Now we have that

$$d\alpha_1 \wedge \alpha_1 \wedge \pi = \pi \wedge (-dz_{n-1} \wedge dz_1 \wedge dz_n) = \pi \wedge (-\alpha_2 \wedge \pi \wedge \alpha_1) = 0,$$

and therefore π r is a linear combination of dz_1, dz_{n-1}, dz_n. Noting that $dz_{n-1} \equiv z_{n-2}dz_1 \mod \alpha_1, \alpha_2$,

$$\begin{aligned} \pi &= adz_1 + bdz_{n-1} + cdz_n, \\ &= adz_1 + bz_{n-2}dz_1 + cz_{n-1}dz_1 \mod \alpha_1, \alpha_2 \end{aligned}$$

where $\psi = a + bz_{n-2} + cz_{n-1}$ is nonzero, since we have assumed that $\pi \neq 0 \mod I$. From the Goursat congruences we have that

$$d\alpha_2 = -\alpha_3 \wedge \pi \mod \alpha_1, \alpha_2,$$

while from equation (4.3) we have

$$d\alpha_2 = -dz_{n-2} \wedge dz_1,$$

and thus

$$-dz_{n-2} \wedge dz_1 = -\alpha_3 \wedge \pi \mod \alpha_1, \alpha_2,$$

which means that

$$\alpha_3 = \lambda(x)dz_{n-2} \mod dz_1, \alpha_1, \alpha_2,$$

for a nonzero function $\lambda(x)$. Therefore we can rewrite this as

$$\alpha_3 = dz_{n-2} - \frac{1}{\lambda(x)}dz_1 \mod dz_1, \alpha_1, \alpha_2,$$

and if we set $z_{n-3} = 1/\lambda(x)$ we have

$$\alpha_3 = dz_{n-2} - z_{n-3}dz_1 \mod \alpha_1, \alpha_2,$$

and we can therefore let

$$\alpha_3 = dz_{n-2} - z_{n-3}dz_1.$$

If we inductively continue this procedure using the Goursat congruences we obtain

$$\begin{aligned} \alpha_4 &= dz_{n-3} - z_{n-4}dz_1, \\ &\vdots \\ \alpha_s &= dz_3 - z_2dz_1. \end{aligned}$$

Now, from the Goursat congruences we have that

$$d\alpha_s \neq 0 \mod I,$$

and therefore

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_s \wedge d\alpha_s \neq 0.$$

f we substitute the α_i in the above expression we obtain en l'anterior expressió obtenim

$$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \neq 0,$$

and therefore the function $z_1, \ldots, z_n$ can serve as a local coordinate system. ■

The following example illustrates the power of the Goursat's theorem by applying it in order to linearize a nonlinear system. Note that the integral curves of a system in Goursat normal form are completely determined by two arbitrary functions in one variable and their derivatives. For example, once $z_1(\tau)$ and $z_s(\tau)$ are known, all of the other coordinates are determined from

$$z_i = \frac{\dot{z}_{i+1}(\tau)}{\dot{z}_i(\tau)},$$

where the dot indicates the standard derivative with respect to the independent variable τ. Because of this property, these two coordinates are sometimes referred to as *linearizing outputs for the Pfaffian system.*

4.2.4 Example (Feedback Linearization by Goursat Normal Form)
Consider the following non linear system with s configuration variables and a singles input

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_s, u), \\
\dot{x}_2 &= f_2(x_1, \ldots, x_s, u), \\
&\vdots \\
\dot{x}_s &= f_s(x_1, \ldots, x_s, u).
\end{aligned}$$

Equivalently, we can look at following Pfaffian system,

$$I = \{dx_i - f_i(x_1, \ldots, x_s, u)dt\}, \quad 1 \leq i \leq s.$$

The system is of codimension 2 since we have s constraints and $s+2$ variables, namely $x_1, \ldots, x_s, u, t$. Assume that the form $\pi = dt$ satisfies the Goursat congruences. Then by Goursat's theorem there exists a coordinate transformation $z = \Phi(x, u, t)$ such that I is generated by

$$I = \{dz_3 - z_2 dz_1,\, dz_4 - z_3 dz_1,\, \ldots,\, dz_{s+2} - z_{s+1} dz_1\}.$$

The annihilating distribution of the above codistribution is

$$\begin{aligned}
\dot{z}_1 &= v_1, \\
\dot{z}_2 &= v_2, \\
\dot{z}_3 &= z_2 v_1, \\
&\vdots \\
\dot{z}_{s+2} &= z_{s+1} v_1,
\end{aligned}$$

which, if we set $v_1 = 1$, is clearly a linear system. If it turns out that the z_1 coordinate corresponds to time in the original coordinates, that is $z_1 = t$, then the connection becomes even more clear. Goursat's theorem can thus be used to linearize single-input nonlinear systems that satisfy the Goursat congruences.

4.3 The Extended Goursat Normal Form

While the Goursat normal form is powerful, it is restricted to Pfaffian systems of codimension two. In order to study Pfaffian systems of higher codimension, we present the extended Goursat normal form. Whereas the Goursat normal form can be thought of as a single chain of integrators, the extended Goursat form consists of many chains of integrators. Consider the following definition:

4.3.1 Definition A Pfaffian system I in $\mathbb{R}^{n+m+1}$ of codimension $m+1$ is in *extended Goursat normal form* if it is generated by n constraints of the form

$$I = \{dz_i^j - z_{i+1}^j dz_0 : i = 1, \ldots, s_j;\ j = 1, \ldots, m\}. \tag{4.4}$$

This is a direct extension of the Goursat normal form, and all integral curves of (4.4) are determined by th $m+1$ functions $z_0(t), z_1^1(t), \ldots, z_1^m(t)$ and their derivatives with respect to the parameter t. The notation has been changed slightly; the canonical constraints are now $dz_i^j - z_{i+1}^j dz_0$, whereas before they were $dz_i - z_{i-1}dz_1$. For the Goursat form, the constraint in the last nontrivial derived system was $dz_n - zn - 1dz_1$; in the extended Goursat normal form, it will be $dz_1^j - z_2^j dz_0$. We refer to the set of constraints with superscript j as the j-th *tower* (the reason for this name will become clear after we compute the derived flag).

Conditions for converting a Pfaffian system to extended Goursat normal form are given by the following theorem:

4.3.2 Theorem *Let I be a Pfaffian system of codimension $m+1$. If (and only if) there exists a set of generators $\{\alpha_i^j : i = 1, \ldots, s_j;\ j = 1, \ldots, m\}$ or I and an integrable one form π such that for all j,*

$$\begin{aligned} d\alpha_i^j &\equiv -\alpha_{i+1}^j \wedge \pi, \qquad \bmod I^{(s_j-i)} \quad i = 1, \ldots, s_j - 1, \\ d\alpha_i^j &\not\equiv 0 \qquad \bmod I, \end{aligned} \tag{4.5}$$

Then existeix then there exists a set of coordinates z such that I is in extended Goursat normal form,

$$I = \{dz_i^j - z_{i+1}^j dz_0 : i = 1, \ldots, s_j;\ j = 1, \ldots, m\}.$$

If the one-form π which satisfies the congruences (4.5) is not integrable, then the Frobenius theorem cannot be used to find the coordinates. In the special case where $s_1 > s_2$, that is, there is one tower which is strictly longer than the others, it can be shown that if there exists any π which satisfies the congruences, then there also exists an integrable π' which also satisfies the congruences (with a rescaling of the basis forms).However, if $s_1 = s_2$, or there are at least two towers which are longest, this is no longer true. Thus, the assumption that π is integrable is necessary for the general case.

If I can be converted to extended Goursat normal form, then the derived flag of I has the structure

$$\begin{array}{rcllllllll}
I & = & \{\alpha_1^1, & \ldots & \ldots & \alpha_{s_1-1}^1, & \alpha_{s_1}^1, & \ldots & \alpha_1^m, & \ldots \quad \alpha_{s_m}^m\}, \\
I^{(1)} & = & \{\alpha_1^1, & \ldots & \ldots & \alpha_{s_1-1}^1, & & \ldots & \alpha_1^m, & \ldots \ \}, \\
 & & \vdots & & \iddots & & & \vdots & \iddots & \\
I^{(s_m-1)} & = & \{\alpha_1^1, & \ldots & \alpha_{s_1-s_m+1}^1, & & & \ldots & \alpha_1^m\}, & \\
 & & \vdots & & \iddots & & & & & \\
I^{(s_1-2)} & = & \{\alpha_1^1, & \alpha_2^1\}, & & & & & & \\
I^{(s_1-1)} & = & \{\alpha_1^1\}, & & & & & & & \\
I^{(s_1)} & = & \{0\}, & & & & & & &
\end{array}$$

where the forms in each level have been arranged to show the different towers. The superscripts j indicate the tower to which each form belongs, and the subscripts i index the position of the form within the j-th tower. There are s_j forms in the j-th tower.

5
Procedures

In this section we will give a series of steps and explanations needed to be followed to find the behavior of the state variables of a given system. In this paper we consider driftless control systems, i.e. systems of the form

$$\dot{x} = \sum_{i=1}^{\overline{m}} g_i u_i, \quad x \in \mathbb{R}^{\overline{n}}$$

called *nonholonomic systems* or *driftless systems*, where $\overline{m}$ is the number of controls and $\overline{n}$ the dimension of the space where we work.

The associated distribution to this type of systems is generated by the vector fields:

$$\Delta = \langle g_1, g_2, \ldots, g_{\overline{m}} \rangle$$

The dual of this distribution is a subspace of the cotangent space defined, in this case, as follows:

$$\Delta^{\perp} = \langle \omega \in \Lambda^1(\mathbb{R}^{\overline{n}}) \,|\, \omega \lrcorner g = 0, \forall g \in \Delta \rangle$$

where the 1-forms have to be linearly independents. Notice that since $\dim\Delta = \overline{m}$, then $\dim\Delta^{\perp} = \overline{n} - \overline{m}$.

By the definition (3.3.1), the associtated Pffafian system to our control systems is

$$\omega_1 = \omega_2 = \ldots = \omega_{\overline{n}-\overline{m}} = 0$$

that is a system of codimension $\overline{m}$.

In the previous chapter we saw how to express the basis elements of the codistribution in the Goursat normal form or in the extended normal Goursat form, when the Pfaffian system is of codimension 2 or greater than 2 respectively. Thus, being following the constructive demonstration of the Pfaffian and Engel's theorems or following the develop theory about the extended Goursat normal form, given a Pfaffian system on $\mathbb{R}^{n+m+1}$ where $\overline{n} = n + m + 1$, we are able to find chains of integrators so that, of the Pfaffian system has codimension $\overline{m} = m + 1$, the ideal generated by the 1-forms belonging on the codistribution is expressed as:

$$I = \{\omega_i^j = dz_i^j - z_{i+1}^j dz_0 : i = 1, \dots, s_j,\ j = 1, \dots, m\}$$

where s_j satisfies $\overline{n} = m + 1 + \sum_{j=1}^{m} s_j$.

Once founded the change of the 1-forms to the extended Goursat normal form we want to seek for $m+1$ generic vector fields expressed as:

$$\overline{g}_k = \left(a_0, a_1^1, \dots, a_{s_1}^1, a_{s_1+1}^1, \dots, a_1^m, \dots, a_{s_m}^m, a_{s_m+1}^m\right), \quad k = 0, \dots, m$$

such that the contraction with all the 1-forms be zero, i.e, for all k:

$$\overline{g}_k \lrcorner \left\{ \begin{array}{c} dz_1^j - z_2^j\, dz_0 \\ dz_2^j - z_3^j\, dz_0 \\ \vdots \\ dz_{s_j}^j - z_{s_j+1}^j\, dz_0 \end{array} \right\} = 0, \quad j = 1, \dots, m$$

A possible solution is g_0 such that:

$$\begin{aligned} a_0 &= 1 \\ a_1^j &= z_2^j \\ &\vdots \\ a_{s_j} &= z_{s_j+1}^j \end{aligned}$$

and

$$g_j = \frac{\partial}{\partial z_{s_j+1}^j}, \quad j = 1, \dots, m$$

So that the system expressed in the new state variables becomes

$$\left\{\begin{array}{ccc}
\dot{z}_0 & = & u_0 \\
\dot{z}_1^1 & = & z_2^1 u_0 \\
 & \vdots & \\
\dot{z}_{s_1}^1 & = & z_{s_1+1}^1 u_0 \\
\dot{z}_{s_1+1}^1 & = & u_1 \\
\dot{z}_1^2 & = & z_2^2 u_0 \\
 & \vdots & \\
\dot{z}_{s_2}^2 & = & z_{s_2+1}^2 u_0 \\
\dot{z}_{s_2+1}^2 & = & u_2 \\
 & \vdots & \\
\dot{z}_1^m & = & z_2^m u_0 \\
 & \vdots & \\
\dot{z}_{s_m}^m & = & z_{s_m+1}^m u_0 \\
\dot{z}_{s_m+1}^m & = & u_m
\end{array}\right. \tag{5.1}$$

we call it *system in the canonical form associated to the Goursat form.*

Often, the system found by doing the contraction of the fields with the 1-forms and the system obtained by derivating the variables $\{z\}$ is not the same. To achieve the last one being in the canonial Goursat form, it should be necessary do a feedback. Finally we will stablish the difeomorphism that matches the state variables $\{x\}$ and $\{z\}$. Notice that sometimes is convenient to add new state variables to achieve the same dimensions.

Is immediate to see that

$$\begin{aligned}
y_0 &= z_0 \\
y_1 &= z_1^1 \\
&\vdots \\
y_m &= z_1^m
\end{aligned}$$

are a finite family of flatoutputs of the system (5.1), because one can express the variables $\{z\}$ depending on the flat outputs and its derivatives, let's see it:

$$\left\{\begin{array}{c}
z_2^j = \frac{\dot{z}_1^j}{u_0} = \frac{\dot{y}_j}{\dot{y}_0} \\
z_3^j = \frac{\dot{z}_2^j}{u_0} = z_3^j(\dot{y}_0, \ddot{y}_0, \dot{y}_j, \ddot{y}_j) \\
\vdots \\
z_{s_j+1}^j = \frac{\dot{z}_{s_j}^j}{u_0} = z_{s_j+1}^j\left(\dot{y}_0, \ldots, y_0^{(s_j)}, \dot{y}_j, \ldots, y_j^{(s_j)}\right)
\end{array}\right.$$

Let $s_0 = \max\{s_1, \ldots, s_m\}$. The number of variables needed to express the variables $\{z\}$ is

$$c_y = 1 + s_0 + \sum_{j=1}^{m}(s_j + 1) = s_0 + \overline{n}$$

To consider the difeomorphism between the new variables and

$$\{y_0, \dot{y}_0, \ldots, y_0^{(s_0)}, y_1, \ldots, y_1^{(s_1)}, \ldots, y_m, \ldots, y_m^{(s_m)}\}$$

We have to consider the prolongation:

$$\begin{aligned} z_1^0 &= u_0 \\ &\vdots \\ z_{s_0}^0 &= u_0^{(s_0-1)} \\ v &= \dot{z}_{s_0}^0 \end{aligned}$$

The goal to be achived in a system given initial and final conditions to the state variables is to find a motor controls that at each instant of time the solution trajectories of the original system pass through c_i and c_f.

We will impose then, the conditions c_i i c_f to the original state variables. Through the difeomorphism $\{x\} \leftrightarrow \{z\}$ we will find the corresponding initial and final conditions for $\{z\}$ that will be denoted by $\overline{c_i}$ and $\overline{c_f}$. With this data and adding conditions to $z_1^0, z_2^0, \ldots, z_{s_0}^0$, we find the conditions that have to be satisfied by the flat outputs and its derivatives thanks to the difeomorphism $\{z\} \leftrightarrow \{y\}$ and that will be denoted by $\hat{c_i}$ i $\hat{c_f}$.

Notice that for each flat output $y_j, \quad j = 1, \ldots, m$ we hav $s_j + 1$ initial and final conditions and 2 initial and final conditions for y_0.

Given $s_j + 1$ initial and final conditions (in total $2(s_j + 1)$ conditions), for all j, there exists a unique polynomial of degree $2s_j + 1$ denoted by $P_{s_j}(t)$ such that:

$$y_j(t) = P_{s_j}(t), \quad j = 1, \ldots, m$$

Imposing the above conditions, the interpolation polynomials are determined and, consequently, the flat outputs expression involving the time is found.

$$y_0(t), y_1(t), \ldots, y_m(t)$$

Clearly, its derivatives will be depend also on time.

We have commented above that the variables $\{z\}$ can be expressed involving the flat outputs and its derivatives that involve the time.

The flat output system becomes:

$$\begin{cases} y_0^{(s_0+1)} = v \\ y_j^{(s_j+1)} = \frac{d^{s_j+1}}{dt^{s_j+1}} z_1^j, \quad j = 1, \ldots, m \end{cases}$$

From here the controls $u_j(t), \quad j = 1, \ldots, m$ will be found. We replace the controls in the original system and we simulate the trajectories.

6
Driftless systems with m inputs and $m+2$ states

In this section we will focus on driftless systems with m inputs and $m+2$ states of the form

$$\dot{x} = \sum_{k=1}^{m} u^k f_k(x), \quad x \in X \subset \mathbb{R}^n$$

This type of systems have been shown to be flat as soon as controllable and can be converted by dynamic feedback and coordinate change into a multi-input chained form:

$$\begin{array}{rcl} \dot{y}_1^1 & = & u^1 \\ \dot{y}_k^1 & = & y_k^2 u^1 \\ & \vdots & \\ \dot{y}_j^{(n_k-1)} & = & y_k^{(n_k)} u^1 \end{array}$$

for $k = 2, \dots, m$.

6.1 Pfaffian systems of 2 equations

6.1.1 Definition A vector field ξ such that $i_\xi I = 0$ and $i_\xi \, dI \subset I$ is called a *characteristic vector field* of I.

6.1.2 Definition Let I be an algebraic ideal. We define the *class* c of I as the number of variables n minus the number of independent characteristic vector fields.

The importance of the retracting space in this section is that I can be written using only $\dim C(I)$ variables instead of n. Therefore, the class c is exactly the dimension of the retracting space.

6.1.3 Definition The *wedge length* of I is the smallest integer ρ such that

$$\forall \alpha \in I, \quad (d\alpha)^{\rho+1} \equiv 0 \mod I$$

Notice that after a suitable coordinate change, I can be rewritten using only c variables, instead of n.

6.1.4 Theorem *Let I be a controllable Pfaffian system of two equations in n variables. Then I can be prolonged into a Goursat system.*

The proof is different according as the dimension n is odd or even. Notice also that since I is generated by two equations, there are only two cases for the rank structure of the derived flag either

$$\begin{aligned}&\dim I^{(1)} = 1, \dim I^{(2)} = 0\\&\dim I^{(1)} = 0\end{aligned}$$

The first case corresponds to a system of class 4. We can apply the Engel's theorem to find that

$$I = \{dx^2 - x^3 dx^1, dx^3 - x^4 dx^1\}$$

and is already in Goursat form. We prove the theorem for the other cases.

6.2 The odd-dimensional case

An important result that we will use is the following lemma

6.2.1 Lemma *Let I be a controllable Pfaffian system of class $n = 2p+1$. Then I contains a form of rank $1 < r < p$.*

Proof:
We will prove the lemma in the odd case. Notice that in the even case the proof is analogous. Let I be spanned by the 1-forms ω_1 and ω_2. The Pfaffian system I is controllable. For that reason, each non-zero 1-form belonging to I is non integrable. Let's see this:

Assume there exists a 1-form ω which is integrable, then $\omega = dh = 0$. Therefore $h = \text{ctt}$ which implies that the trajectories of the system are restricted to the manifold $h = \text{ctt}$ and I will be non controllable.

The property of the Frobenius theorem is not satisfied. Therefore, $d\alpha \wedge \alpha \neq 0$ and α is of rank at least one.

Notice that the wedge length ρ has to be less or equal than $p-1$, because if $\rho = p$, by defnition of wedge length

$$\forall \alpha \in I, \quad (d\alpha)^{p+1} \equiv 0 \mod I$$

and $(d\alpha)^{p+1}$ is a $2p+2$-form in a space of $2p+1$. Therefore $\rho \leq p-1$. We distinguish different cases.

- $\rho \leq p-2$
  If the wedge length is $\rho \leq p-2$ every non-zero $\omega \in I$ does the job. Indeed, there are forms $\omega_3, \ldots, \omega_{\rho+2}$ such that $\omega_1 \wedge \ldots \wedge \omega_{\rho+2} \neq 0$ and

$$d\omega = \sum_{k=1}^{\rho+2} \omega_k \wedge \nu^k$$

for some forms $\nu^1, \dots, \nu^{\rho+2}$.

Thus, if $\rho < p-2$:

$$(d\omega)^p = \left(\sum_{k=1}^{\rho+2} \omega_k \wedge \nu^k\right)^p = 0, \quad \rho+2 < p$$

and if $\rho = p-2$:

$$(d\omega)^p = \left(\sum_{k=1}^{\rho+2} \omega_k \wedge \nu^k\right)^p = \lambda\, (\omega_1 \wedge \nu_1 \wedge \dots \wedge \omega_p \wedge \nu_p)$$

Hence in both cases $(d\omega)^p \wedge \omega = 0$.

- $\rho = p-1$

We thus assume I has maximum wedge length $p-1$. We look for a function λ such that $\omega := \omega^1 + \lambda\omega^2$ satisfies

$$0 = (d\omega)^p \wedge \omega = (d\omega_1 + \lambda\omega_2)^p \wedge (\omega_1 + \lambda\omega_2) + p\, d\lambda \wedge (d\omega_1 + \lambda\, d\omega_2)^{p-1} \wedge \omega_2 \wedge \omega_1$$

But $(d\omega)^p \wedge \omega$ is a monomial. Hence $(d\omega)^p \wedge \omega = 0$ is nothing but a scalar equation of the form

$$d\lambda \left(\sum_{i=0}^{p-1} \lambda^i f_i(x)\right) + \sum_{j=0}^{p+1} \lambda^j a_j(x) = 0 \tag{6.1}$$

where the a_j's are functions obtained by expanding $(d\omega_1 + \lambda\, d\omega_2)^p \wedge (\omega_1 + \lambda\omega_2)$ and the f_i's are vectors made from the components (up to constant coefficients) on the basis $\{\, dx^{i_1} \wedge \dots \wedge dx^{i_{2p}}, 1 \leq i_1 < \dots < i_{2p} \leq n\}$ of the expansion of the $2p$-form $(d\omega_1 + \lambda\, d\omega_2)^{p-1} \wedge \omega_2 \wedge \omega_1$.

Since the wedge length is $p-1$, at least one of the f_i's is not zero (by definition of the wedge length at least one $(p-1)$-fold product $(d\omega_1)^i \wedge (d\omega_2)^{p-1-i} \not\equiv 0 \mod I$). In this case (6.1) is a first-order quasilinear partial differential equation which always admits a solution λ for every suitable initial data $\lambda(x_0)$.

Then, p is the first integer such that $(d\alpha)^p \wedge \alpha = 0$ and α has rank $1 < r < p$. ∎

We are now in position to prove the theorem when I has odd class $n = 2p+1$. Suppose $I = \langle \omega_1, \omega_2 \rangle$, where

$$\begin{aligned} \omega_1 &= b_1\, dx^1 + \dots + b_n\, dx^n \\ \omega_2 &= c_1\, dx^1 + \dots + c_n\, dx^n \end{aligned}$$

and have to be linearly independent. By lemma 6.2.1, I contains a form of rank $1 < r < p$, hence is spanned in suitable coordinates by:

$$\begin{aligned} \overline{\omega}_1 &= dx^2 - \textstyle\sum_{k=1}^{r} x^{2k+1}\, dx^{2k+2} \\ \overline{\omega}_2 &= a_1\, dx^1 + \textstyle\sum_{k=3}^{n} a_k\, dx^k \end{aligned} \tag{6.2}$$

where the first equality is obtained thanks to the constructive Pfaffian theorem and the second one doing the following:

$$\overline{\omega}_2 = \omega_2 - c_2\overline{\omega}_1 = c_1\,dx^1 + c_2\,dx^2 + \ldots + c_n\,dx^n - c_2\,dx^2 + \sum_{k=1}^{r} c_2 x^{2k+1}\,dx^{2k+2}$$

setting $a_i = c_i$ when i is odd and $a_i = c_i + c_2 x^{i-1}$ when i is even and $i \geq 4$. Therefore, we obtain

$$\overline{\omega}_2 = a_1\,dx^1 + \sum_{k=3}^{n} a_k\,dx^k$$

where the a_k's are functions of x. We distinguish two subcases: either the functions $a_1, a_{2r+3}, \ldots, a_n$ are all zero or one of those functions (suppose a_1) is not zero. Of course, at least two of a_k are not zero because I, being controllable, contains no integrable forms.

Notice that from now on, we won't need the class to be exactly $2p+1$. Class at most $2r+3$ is enough. Let's see this:

Since $r < p$, that is the same as $r \leq p-1$ which implies that $2r+3 \leq 2p+1$, thus, the generators of the associated Pfaffian system can be expressed in at most $2r+3$ coordinates instead of $n = 2p+1$.

From here, we distinguish two cases:

6.2.1 Case $a_1 = a_{2r+3} = \ldots = a_n = 0$

Since at least two a_k are different from zero, up to a permutation we can assume that $a_3 \neq 0$ and $a_4 \neq 0$. Hence we may assume that I is spanned by

$$\begin{aligned} \alpha_1 &= dx^2 - \textstyle\sum_{k=1}^{r} x^{2k+1}\,dx^{2k+2} \\ \alpha_2 &= dx^3 - \textstyle\sum_{k=4}^{2r+2} a_k\,dx^k \end{aligned}$$

Now, we can prolong I by adding the forms

$$\left.\begin{aligned} \alpha_{k,1} &= dx^{2k+1} - \xi^{2k+1}\,dx^4 \\ \alpha_{k,2} &= dx^{2k+2} - \xi^{2k+2}\,dx^4 \\ \alpha_{k,3} &= d\xi^{2k+2} - \zeta^{2k+2}\,dx^4 \end{aligned}\right\}, \quad k = 2, \ldots, r \tag{6.3}$$

We call the prolonged system J and is spanned by (6.3) and

$$\begin{aligned} \widetilde{\alpha}_1 &= dx^2 - \left(x^3 - \sum_{k=2}^{r} x^{2k+1}\xi^{2k+2}\right) dx^4 \\ \widetilde{\alpha}_2 &= dx^3 - \left(a_4 + \sum_{k=5}^{2r+2} a_k\xi^k\right) dx^4 \end{aligned}$$

That are our original 1-forms α_1 and α_2 after a change of coordinates using (6.3).

Notice that since we use n variables to describe I then the ideal is of class n, therefore, at least one of the functions a_k's depends on x_1. We can do the following change of coordinates:

$$\begin{aligned}\widetilde{x}_1 &:= a_4 + \sum_{k=5}^{2r+2} a_k \xi^k + \sum_{l=2}^{r} \left(\xi^{2l+1}\xi^{2l+2} + x^{2l+1}\zeta^{2l+2}\right)\\ \widetilde{x}_3 &:= x^3 + \sum_{k=2}^{r} x^{2k+1}\xi^{2k+2}\end{aligned}$$

and leaving the others variables unchanged. Take into account that

$$d\widetilde{x}^3 \equiv dx^3 + \sum_{k=2}^{r} \left(\xi^{2k+1}\xi^{2k+2} + x^{2k+1}\zeta^{2k+2}\right) dx^4 \qquad \text{mod } \alpha_{k,1}, \alpha_{k,2}, \alpha_{k,3}, k = 2, \ldots, r$$

The prolonged system J becomes spanned by

$$\begin{aligned}\overline{\alpha}_1 &= dx^2 - \widetilde{x}^3\, dx^4\\ \overline{\alpha}_2 &= d\widetilde{x}^3 - \widetilde{x}^1\, dx^4\end{aligned}$$

and $\alpha_{k,1}, \alpha_{k,2}, \alpha_{k,3}$, hence is in Goursat form. Let's write the chains

$$\begin{array}{l}\overline{\alpha}_1 = dx^2 - \widetilde{x}^3\, dx^4\\ \overline{\alpha}_2 = d\widetilde{x}^3 - \widetilde{x}^1\, dx^4\\ \hline \alpha_{2,2} = dx^6 - \xi^6\, dx^4\\ \vdots\\ \alpha_{r,2} = dx^{2r+2} - \xi^{2r+2}\, dx^4\\ \alpha_{2,3} = d\xi^6 - \zeta^6\, dx^4\\ \vdots\\ \alpha_{r,3} = d\xi^{2r+2} - \zeta^{2r+2}\, dx^4\\ \hline \alpha_{2,1} = dx^5 - \xi^5\, dx^4\\ \vdots\\ \alpha_{r,1} = dx^{2r+1} - \xi^{2r+1}\, dx^4\end{array}$$

6.2.2 Case $a_1 \neq 0$

Recall the forms (6.7). Since $a_1 \neq 0$ we can divide all the terms of $\overline{\omega}_2$ by a_1, being $\overline{a}_k = a_k/a_1$then I is spanned by

$$\omega_1 := dx^2 - \sum_{k=1}^{r} x^{2k+1}\, dx^{2k+2} \tag{6.4}$$

$$\omega_2 := dx^1 - \sum_{k=3}^{n} \overline{a}_k\, dx^k \tag{6.5}$$

We perfom the following coordinate change

$$\begin{aligned} z^1 &:= g(x) \\ z^i &:= x^i, \qquad i = 2, \dots, n \end{aligned}$$

where g is a function that satisfies $\partial_1 g \neq 0$ and has to be determined in order to zero the coefficient on dz^n. Thus:

$$dz^1 = \partial_1 g \, dx^1 + \cdots + \partial_n g \, dx^n$$

Using (6.5) we get

$$dz^1 = \partial_1 g \omega_1 g \left(\sum_{k=3}^{n} a_k \, dx^k \right) + \partial_2 g \, dx^2 + \cdots + \partial_n g \, dx^n$$

$$dz^1 = \partial_1 g \omega^2 + \partial_2 g \, dx^2 + (a_3 \partial_1 g + \partial_3 g) \, dx^3 + \cdots + (a_n \partial_1 g + \partial_n g) \, dx^n$$

Now, usgin (6.4) the previous expression becomes

$$dz^1 - \partial_1 g \omega^2 - (a_n \partial_1 g + \partial_n g) \, dx^n = \partial_2 g \left(\omega^1 + \sum_{k=1}^{r} x^{2k+1} \, dx^{2k+2} \right) + \sum_{k=3}^{n-1} (a_k \partial_1 g + \partial_k g) \, dx^k$$

Therefore,

$$dz^1 = \partial_1 g \omega^2 + (a_n \partial_1 g + \partial_n g) \, dx^n, \qquad \text{mod } (\omega^1, \, dx^3, \dots, \, dx^{n-1})$$

Choosing for g a solution of the first-order linear partial differential equation

$$\partial_n g + a_n \partial_1 g = 0$$

because we want to zero the dz^n coefficient such that $\partial_1 g \neq 0$. Then I is clearly spanned by

$$\begin{aligned} \overline{\omega}_1 &= dz^2 - \sum_{k=1}^{r} z^{2k+1} \, dz^{2k+2} \\ \overline{\omega}_2 &= dz^2 - \sum_{k=3}^{n-1} b_k \, dz^k \end{aligned}$$

Where b_k are functions.

Now prolong I by adding the forms $\omega_3 = dz^3 - \xi^3 \, dz^4$ and

$$\left.\begin{aligned} \omega_{k,1} &= dz^{2k+1} - \xi^{2k+1}\, dz^4 \\ \omega_{k,2} &= dz^{2k+2} - \xi^{2k+2}\, dz^4 \\ \omega_{k,3} &= d\xi^{2k+2} - \zeta^{2k+2}\, dz^4 \end{aligned}\right\}, \quad k = 2, \ldots, p-1 \tag{6.6}$$

We call the prolonged system J and is spanned by (6.6) and

$$\begin{aligned} \widetilde{\omega}_1 &= dz^1 - (b_3\xi^3 + b_4 + \sum_{k=5}^{n-1} b_k \xi^k)\, dz^4 \\ \widetilde{\omega}_2 &= dz^2 - (z^3 + \sum_{k=2}^{r} z^{2k+1}\xi^{2k+2})\, dz^4 \\ \omega_3 &= dz^3 - \xi^3\, dz^4 \end{aligned}$$

We remark that I is of class n, hence, at least one of the functions b_k depends on z^n. We can do the following change of coordinates:

$$\begin{aligned} \widetilde{z}^3 &:= z^3 + \sum_{k=2}^{r} z^{2k+1}\xi^{2k+2} \\ \widetilde{z}^n &:= b_3\xi^3 + b_4 + \sum_{k=5}^{n} b_k \xi^k \\ \widetilde{\xi}^3 &:= \xi^3 + \sum_{k=2}^{r} (\xi^{2k+1}\xi^{2k+2} + z^{2k+1}\zeta^{2k+2}) \end{aligned}$$

and leaving the other variables unchanged. Noticing that

$$d\widetilde{z}^3 \equiv dz^3 + \sum_{k=2}^{r} (\xi^{2k+1}\xi^{2k+2} + z^{2k+1}\zeta^{2k+2})\, dz^4 \bmod (6.6)$$

Now the ideal J is spanned by

$$\begin{aligned} \overline{\omega}_1 &= dz^1 - \widetilde{z}^n\, dz^4 \\ \overline{\omega}_2 &= dz^2 - \widetilde{z}^3\, dz^4 \\ \overline{\omega}_3 &= d\widetilde{z}^3 - \widetilde{\xi}^3\, dz^4 \end{aligned}$$

and (6.6). Therefore, is in Goursat normal form. Let's write the chains

$$
\begin{array}{l}
\overline{\omega}_2 = dz^2 - \widetilde{z}^3\, dz^4 \\
\overline{\omega}_3 = d\widetilde{z}^3 - \widetilde{\xi}_3\, dz^4 \\
\hline
\overline{\omega}_1 = dz^1 - \widetilde{z}^n\, dz^4 \\
\hline
\omega_{2,2} = dz^6 - \xi^6\, dz^4 \\
\vdots \\
\omega_{p-1,2} = dz^{2(p-1)+2} - \xi^{2(p-2)+2}\, dz^4 \\
\omega_{2,3} = d\xi^6 - \zeta^6\, dz^4 \\
\vdots \\
\omega_{p-1,3} = d\xi^{2(p-1)+2} - \zeta^{2(p-1)+2}\, dz^4 \\
\hline
\omega_{2,1} = dz^5 - \xi^5\, dz^4 \\
\vdots \\
\omega_{p-1,1} = dz^{2(p-1)+1} - \xi^{2(p-1)+1}\, dz^4
\end{array}
$$

6.3 The even-dimensional case

In this section we will prove the theorem when I has even class, i.e $n = 2p$. Clearly every form in I has rank $r \leq p-1$ .Suppose $I = \langle \omega_1, \omega_2 \rangle$, where

$$
\begin{aligned}
\omega_1 &= b_1\, dx^1 + \cdots + b_n\, dx^n \\
\omega_2 &= c_1\, dx^1 + \cdots + c_n\, dx^n
\end{aligned}
$$

and have to be linearly independent. I is spanned in a suitable coordinates as

$$
\begin{array}{l}
\overline{\omega}_1 = dx^1 - \sum_{k=1}^{r} x^{2k}\, dx^{2k+1} \\
\overline{\omega}_2 = \sum_{k=2}^{2p} a_k\, dx^k
\end{array}
\tag{6.7}
$$

where the first equality is obtained thanks to the constructive Pfaffian theorem and the second one doing the following:

$$
\overline{\omega}_2 = \omega_2 - c_1\overline{\omega}_1 = c_1\, dx^1 + c_2\, dx^2 + \ldots + c_n\, dx^n - c_1\, dx^1 + \sum_{k=1}^{r} c_1 x^{2k}\, dx^{2k+1}
$$

setting $a_i = c_i$ when i is even and $a_i = c_i + c_1 x^{i-1}$ when i is odd. Therefore, we obtain

$$
\overline{\omega}_2 = \sum_{k=2}^{2p} a_k\, dx^k
$$

For the following cases the theorem is already proved:

- If $r < p-1$, the proof used in the odd-dimensional case applies. Indeed using the following coordinate change:

$$
\begin{aligned}
x^1 &= z^2 \\
x^{2k} &= z^{2k+1} \\
x^{2k+1} &= z^{2k+2} \\
x^{2r+2} &= z^1 \\
x^{2r+3} &= z^{2r+3} \\
&\vdots \\
x^n &= z^n
\end{aligned}
$$

  for $k = 1, \ldots, r$. The 1-forms can be expressed as in (6.7).

- If $r = p-1$ and $a_{2p} = 0$. In this case I is spanned by

$$
\begin{aligned}
\alpha_1 &= dx^1 - \sum_{k=1}^{p-1} x^{2k}\, dx^{2k+1} \\
\alpha_2 &= \sum_{k=2}^{2p-1} a_k\, dx^k = \sum_{k=2}^{2r+1} a_k\, dx^k
\end{aligned}
$$

  Doing the change of coordinates $\overline{x}^{i+1} = x^i, \quad i = 1, \ldots, 2p$ the 1-forms become

$$
\begin{aligned}
\overline{\alpha}_1 &= d\overline{x}^2 - \sum_{k=1}^{r} \overline{x}^{2k+1}\, dx^{2k+2} \\
\overline{\alpha}_2 &= \sum_{k=3}^{2r+2} \overline{a}_k\, d\overline{x}^k
\end{aligned}
$$

  This expressions are the same as in (6.2.1). The proof used there applies.

It remains to show the case when $r = p-1$ and $a_{2p} \neq 0$.

We may assume that I is spanned by

$$
\begin{aligned}
\alpha_1 &= dx^1 - \sum_{k=1}^{p-1} x^{2k}\, dx^{2k+1} \\
\alpha_2 &= dx^2 - \sum_{k=3}^{2p} a_k\, dx^k
\end{aligned}
$$

where since at least two a_k's are not zero, we can assume that $a_2 \neq 0$ and $a_{2p} \neq 0$. We can divide the expression by a_2 and rename the functions a_k. We prolong our system I adding the 1-form

$$\alpha_3 = dx^3 - x^{2p+1}\, dx^{2p-1}$$

Notice that we have introduce a new variable x^{2p+1}. The following lemma proves that that the class of I is the new space dimension.

6.3.1 Lemma *The prolonged system $J := I + \langle \alpha_3 \rangle$ is of class $2p+1$.*

Proof:
We must show that the unique characteristic vector field belonging on J is $\xi = 0$. We consider a generic vector field:

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{2p+1})$$

We impose that $i_\xi \alpha_k = 0$ and $i_\xi\, d\alpha_k \in J$ for $k = 1, 2, 3$. For the first three condition we obtain that:

$$\begin{aligned}
\xi \lrcorner \alpha_1 &= \xi_1 - x^2\xi_3 - \cdots - x^{2p-2}\xi_{2p-2} = 0 \\
\xi \lrcorner \alpha_2 &= \xi_2 - a_3\xi_3 - \cdots - a_{2p}\xi_{2p} = 0 \\
\xi \lrcorner \alpha_3 &= \xi_3 - x^{2p+1}\xi_{2p-1} = 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\xi_1 &= \sum_{k=1}^{p-1} x^{2k}\xi_{2k+1} \\
\xi_2 &= \sum_{k=3}^{2p} a_k\xi_k \\
\xi_3 &= x^{2p+1}\xi_{2p-1}
\end{aligned}$$

On the other hand, imposing $\xi \lrcorner\, d\alpha_3 \in J$ we get:

$$i_\xi\, d\alpha_3 = -\xi_{2p+1}\, dx^{2p-1} + \xi_{2p-1}\, dx^{2p+1} \in J$$

which implies that

$$\xi_{2p+1} = \xi_{2p-1} = 0$$

Now we calculate the exterior derivative of α_1:

$$
\begin{aligned}
d\alpha_1 &= \sum_{k=1}^{p-1} dx^{2k} \wedge dx^{2k+1} = dx^3 \wedge dx^2 + \sum_{k=2}^{p-1} dx^{2k} \wedge dx^{2k+1} \\
&\equiv x^{2p+1}\, dx^{2p-1} \wedge \left(\sum_{k=4}^{2p} a_k\, dx^k \right) + \sum_{k=2}^{p-1} dx^{2k} \wedge dx^{2k+1} \mod J
\end{aligned}
$$

Using $\xi^{2p-1} = 0$ and $i_\xi\, d\alpha_1 \in J$ we get

$$
\left(\xi^{2p-2} + x^{2p-1} \sum_{k=4}^{2p} a_k \xi_k \right) dx^{2p-1} + \sum_{k=2}^{p-2} \left(\xi_{2k}\, dx^{2k+1} - \xi_{2k+1}\, dx^{2k} \right) = 0
$$

Since a_k's do not depend on the variable x^{2p+1}, we deduce that $\xi_4 = \cdots = \xi_{2p-3} = 0$ and $\sum_{k=4}^{2p} a_k \xi^k = 0$. Together with the previous definitions of each component of ξ and the fact that $a_{2p} \neq 0$ we deduce that $\xi = 0$. ■

Thus, J is spanned by

$$
\alpha_1 = dx^1 - \sum_{k=1}^{p-1} x^{2k}\, dx^{2k+1} \tag{6.8}
$$

$$
\alpha_2 = dx^2 - \sum_{k=3}^{2p} a_k\, dx^k \tag{6.9}
$$

$$
\alpha_3 = dx^3 - x^{2p+1}\, dx^{2p-1} \tag{6.10}
$$

A change of coordinates is perfomed in order to modify the generators of J:

$$
\begin{aligned}
z^{2p-2} &:= x^{2p-2} + x^2 x^{2p+1} \\
z^i &:= x^i, \quad i \neq 2p-2
\end{aligned}
$$

Is clear that α_3 in the new variables becomes:

$$
\alpha_3 = dz^3 - z^{2p+1}\, dz^{2p-1}
$$

From here

$$
\alpha_3 - dz^3 = -z^{2p+1}\, dz^{2p-1} \tag{6.11}
$$

For α_2 we have that

$$
\alpha_2 = dz^2 - a_3\, dz^3 - \sum_{k=4}^{2p} a_k\, dz^k \equiv dz^2 - \sum_{k=4}^{2p} b_k\, dz^k \mod J
$$

Finally, for α_1:

$$\alpha_1 = dx_1 - \sum_{k=1}^{p-2} x^{2k}\, dx^{2k+1} - x^{2p-2}\, dx^{2p-1}$$

Using the coordinates change, the last term can be expressed as:

$$x^{2p-2}\, dx^{2p-1} = (z^{2p-2} - z^2 z^{2p+1})\, dz^{2p-1} = z^{2p-2}\, dz^{2p-1} - z^2 z^{2p+1}\, dz^{p-1}$$

So, α_1 in the new coordinates becomes:

$$\alpha_1 = dz^1 - z^2\, dz^3 - \sum_{k=2}^{p-2} z^{2k}\, dz^{2k+1} - z^{2p-2}\, dz^{2p-1} + z^2 z^{2p-1}\, dz^{2p-1}$$

Using (6.11):

$$\begin{aligned}
\alpha_1 &= dz^1 - z^2\, dz^3 - \sum_{k=2}^{p-2} z^{2k}\, dz^{2k+1} - z^{2p-2}\, dz^{2p-1} - z^2 \alpha_3 + z^2\, dz^3 \\
&\equiv dz^1 - \sum_{k=2}^{p-1} z^{2k}\, dz^{2k+1} \mod J
\end{aligned}$$

Hence, the generators of J are:

$$\begin{aligned}
\alpha_1 &= dz^1 - \sum_{k=2}^{p-1} z^{2k}\, dz^{2k+1} \\
\alpha_2 &= dz^2 - \sum_{k=4}^{2p} b_k\, dz^k \\
\alpha_3 &= dz^3 - z^{2p+1}\, dz^{2p-1}
\end{aligned}$$

A change of coordinates is perfomed to zero the coefficient dx^{2p}:

$$\begin{aligned}
x^2 &:= g(z) \\
x^i &:= z^i, \quad i \neq 2
\end{aligned}$$

We proceed as in the previous section

$$dx^2 = \partial_1 g\, dx^1 + \partial_2 g\, dx^2 + \ldots + \partial_{2p+1} g\, dz^{2p+1}$$

Using (6.8), (6.9) and (6.10) the above expression becomes:

$$
\begin{aligned}
dx^2 = {} & \partial_1 g\alpha_1 + \partial_2 g\alpha_2 + \partial_3 g\alpha_3 + (\partial_2 g b_4 + \partial_4 g)\, dz^4 + \sum_{k=5}^{2p-2} (\partial_2 g b_k + z^{k-1}\partial_1 g + a_k g)\, dz^k + \\
& + (\partial_2 g b_{2p-1} + \partial_{2p-1} g + z^{2p-1}\partial_1 g + \partial_3 g z^{2p+1})\, dz^{2p-1} + (\partial_2 g b_{2p} + \partial_{2p} g)\, dz^{2p} + \\
& + (\partial_2 g b_{2p+1} + \partial_{2p+1} g)\, dz^{2p+1}
\end{aligned}
$$

Thus,

$$
dx^2 \equiv \partial_2 g\alpha_2 + (\partial_2 g b_{2p} + \partial_{2p} g)\, dz^{2p} \mod (\alpha_1, \alpha_3, dz^4, \ldots, dz^{2p-1}, dz^{2p+1})
$$

With this change of coordinates, the generators of J become

$$
\begin{aligned}
\overline{\alpha}_1 &= dx^1 - \sum_{k=2}^{p-1} x^{2k}\, dx^{2k+1} \\
\overline{\alpha}_2 &= dx^2 - \sum_{k=4}^{2p-1} b_k\, dx^k - b_{2p+1}\, dx^{2p+1} \\
\overline{\alpha}_3 &= dx^3 - x^{2p+1}\, dx^{2p-1}
\end{aligned}
$$

Now prolong J by $\alpha_4 = dx^{2p-1} - \xi^{2p-2}\, dx^{2p-1}$ and

$$
\left.
\begin{aligned}
\alpha_{k,1} &= dx^{2k} - \xi^{2k}\, dx^{2p-1} \\
\alpha_{k,2} &= dx^{2k+1} - \xi^{2k+1}\, dx^{2p-1} \\
\alpha_{k,3} &= d\xi^{2k+1} - \zeta^{2k+1}\, dx^{2p-1} \\
\alpha_{k,4} &= dx^{2p+2} - \xi^{2p+1}\, dx^{2p-1}
\end{aligned}
\right\}, \quad k = 2, \ldots, p-2 \tag{6.12}
$$

The new prolonged system is spanned by using the previous 1-forms as follows, setting $\xi^3 := x^{2p+1}$

$$
\begin{aligned}
\widetilde{\alpha}_1 &= dx^1 - (x^{2p-2} + \sum_{k=2}^{p-2} x^{2k}\xi^{2k+1})\, dx^{2p-1} \\
\widetilde{\alpha}_2 &= dx^2 - (b_{2p-1} + b_{2p+1}\xi^{2p+1} + \sum_{k=4}^{2p-2} b_k\xi^k)\, dx^{2p-1} \\
\widetilde{\alpha}_3 &= dx^3 - \xi^3\, dx^{2p-1} \\
\widetilde{\alpha}_4 &= d\widetilde{x}^{2p-2} - \widetilde{\xi}^{2p-2}\, dx^{2p-1}
\end{aligned}
$$

and (6.12).

Notice that since we use n variables to describe J then the ideal is of class $2p+1$, therefore, at least one of the functions b_k's depends on x^{2p}. We can do the following change of coordinates:

$$
\begin{aligned}
\widetilde{x}^{2p-2} &:= b_{2p+1}\xi^{2p+1} + b_{2p-1} + \sum_{k=4}^{2p-2} b_k \xi^k + \sum_{l=2}^{p-1}(\xi^{2l}\xi^{2l+1} + x^{2l}\zeta^{2l+1}) \\
\widetilde{x}^{2p} &:= x^{2p-2} + \sum_{k=2}^{p-2} x^{2k}\xi^{2k+1} \\
\widetilde{\xi}^{2p-2} &:= \xi^{2p-2} + \sum_{k=2}^{p-1}(\xi^{2k}\xi^{2k+1} + x^{2k}\zeta^{2k+1})
\end{aligned}
$$

and leaving the others variables unchanged. Take into account that

$$
d\widetilde{x}^{2p-2} \equiv dx^{2p-2} + \sum_{k=2}^{p-1}(\xi^{2k}\xi^{2k+1} + x^{2k}\zeta^{2k1})\, dx^{2p-1} \mod (6.12)
$$

Hence, the system is spanned by

$$
\begin{aligned}
\hat{\alpha}_1 &= dx^1 - \widetilde{x}^{2p}\, dx^{2p-1} \\
\hat{\alpha}_2 &= dx^2 - \widetilde{x}^{2p-2}\, dx^{2p-1} \\
\hat{\alpha}_3 &= dx^3 - \xi^3\, dx^{2p-1} \\
\hat{\alpha}_4 &= d\widetilde{x}^{2p-2} - \widetilde{\xi}^{2p-2}\, dx^{2p-1}
\end{aligned}
$$

and (6.12), hence is in Goursat form. Let's see the chain:

$$
\begin{array}{l}
\hat{\alpha}_2 = dx^2 - \widetilde{x}^{2p-2}\, dx^{2p-1} \\
\hat{\alpha}_4 = d\widetilde{x}^{2p-2} - \widetilde{\xi}^{2p-2}\, dx^{2p-1} \\
\hline
\hat{\alpha}_1 = dx^1 - \widetilde{x}^{2p}\, dx^{2p-1} \\
\hline
\alpha_{2,1} = dx^4 - \xi^4\, dx^{2p-1} \\
\vdots \\
\alpha_{p-2,1} = dx^{2(p-2)} - \xi^{2(p-2)}\, dx^{2p-1} \\
\hline
\alpha_{2,2} = dx^5 - \xi^5\, dx^{2p-1} \\
\vdots \\
\alpha_{p-2,2} = dx^{2(p-2)+1} - \xi^{2(p-2)+1}\, dx^{2p-1} \\
\alpha_{2,3} = d\xi^5 - \zeta^5\, dx^{2p-1} \\
\vdots \\
\alpha_{p-2,3} = d\xi^{2(p-2)+1} - \zeta^{2(p-2)+1}\, dx^{2p-1} \\
\hline
\hat{\alpha}_3 = dx^3 - \xi^3\, dx^{2p-1} \\
\hline
\alpha_{k,4} = dx^{2p+2} - \xi^{2p+1}\, dx^{2p-1}
\end{array}
$$

6.3.1 Application to a driftless system with 7 inputs and 9 states

Notice that we are in the odd-case. Since $n = 2p+1$, then $p = 4$. By the lemma 6.2.1, I contains a form of rank $1 < r < 4$. Hence r could be 2 or 3. The procedures will be focused on $r = 3$. We first analize the case where $a_1 \neq 0$.

Consider that the driftless system after a first static feedback and coordinate change becomes:

$$\begin{cases} \dot{x}^1 = \sum_{k=3}^{8} u^{k-1} b_k(x) \\ \dot{x}^2 = x^3u^2 + x^5u^4 + x^7u^6 \\ \dot{x}^3 = u^1 \\ \dot{x}^4 = u^2 \\ \dot{x}^5 = u^3 \\ \dot{x}^6 = u^4 \\ \dot{x}^7 = u^5 \\ \dot{x}^8 = u^6 \\ \dot{x}^9 = u^7 \end{cases} \tag{6.13}$$

The vector fields of the distribution vanishes the forms belonging on I of the form:

$$\alpha_1 = dx^2 - \sum_{k=1}^{3} x^{2k+1}\, dx^{2k+1}$$
$$\alpha_2 = dx^1 - \sum_{k=3}^{8} b_k(x)\, dx^k$$

Now prolong I by adding the forms $\alpha_3 = dx^3 - \xi^3\, dx^4$ and

$$\begin{aligned} \alpha_4 &= dx^5 - \xi^5\, dx^4 \\ \alpha_5 &= dx^6 - \xi^6\, dx^4 \\ \alpha_6 &= dx^7 - \xi^7\, dx^4 \\ \alpha_7 &= dx^8 - \xi^8\, dx^4 \\ \alpha_8 &= d\xi^6 - \zeta^6\, dx^4 \\ \alpha_9 &= d\xi^8 - \zeta^8\, dx^4 \end{aligned}$$

We call the new ideal $J := I + \{\alpha_3, \ldots, \alpha_9\}$. Written in a more usual way:

$$\begin{aligned} u^1 &= \xi^3 v^2 \\ u^2 &= v^2 \\ u^3 &= \xi^5 v^2 \\ u^4 &= \xi^6 v^2 \\ u^5 &= \xi^7 v^2 \\ u^6 &= \xi^8 v^2 \end{aligned}$$

$$
\begin{aligned}
\dot{\xi}^3 &= v^1 \\
\dot{\xi}^5 &= v^3 \\
\dot{\xi}^6 &= \zeta^6 v^2 \\
\dot{\zeta}^6 &= v^4 \\
\dot{\xi}^7 &= v^5 \\
\dot{\xi}^8 &= \zeta^8 v^2 \\
\dot{\zeta}^8 &= v^6
\end{aligned}
$$

The prolongued system J is spanned by:

$$
\begin{aligned}
\alpha_1 &= dx^1 - (b_3\xi^3 + b_4 + \sum_{k=5}^{9} b_k \xi^k)\, dx^4 \\
\alpha_2 &= dx^2 - \left(x^3 + \sum_{k=2}^{r} x^{2k+1}\xi^{2k+2} \right) dx^4 \\
\alpha_3 &= dx^3 - \xi^3\, dx^4
\end{aligned}
$$

and $\alpha_4, \ldots, \alpha_9$

Changing coordinates by

$$
\begin{aligned}
\widetilde{x}^3 &:= x^3 + \sum_{k=2}^{r} x^{2k+1}\xi^{2k+2} \\
\widetilde{x}^9 &:= b_3\xi^3 + b_4 + \sum_{k=5}^{9} b_k \xi^k \\
\widetilde{\xi}^3 &:= \xi^3 + \sum_{k=2}^{3} \left(\xi^{2k+1}\xi^{2k+2} + x^{2k+1}\zeta^{2k+2} \right)
\end{aligned}
$$

leaving the other variables unchanged and noticing that

$$
d\widetilde{x}^3 \equiv dx^3 + \sum_{k=2}^{3} \left(\xi^{2k+1}\xi^{2k+2} + x^{2k+1}\zeta^{2k+2} \right) dx^4, \qquad \mod \alpha_4, \ldots, \alpha_9
$$

we see that J is spanned by

$$
\begin{aligned}
\overline{\alpha}_1 &= dx^1 - \widetilde{x}^9\, dx^4 \\
\overline{\alpha}_2 &= dx^2 - \widetilde{x}^3\, dx^4 \\
\overline{\alpha}_3 &= d\widetilde{x}^3 - \widetilde{x}^3\, dx^4
\end{aligned}
$$

and $\alpha_4, \ldots, \alpha_9$. The forms belonging on J vanishes the vector fields of the codistribution of the following system:

$$\left\{\begin{aligned}
\dot{x}^1 &= \left(\xi^3 b_1 + b_2 + b_3\xi^5 + b_4\xi^6 + b_5\xi^7 + b_6\xi^8\right) v^2 \\
\dot{x}^2 &= \left(x^3 + \sum_{k=1}^{3} x^{2k+1}\xi^{2k+2}\right) v^2 \\
\dot{x}^3 &= \xi^3 v^2 \\
\dot{x}^4 &= v^2 \\
\dot{x}^5 &= \xi^5 v^2 \\
\dot{x}^6 &= \xi^6 v^2 \\
\dot{x}^7 &= \xi^7 v^2 \\
\dot{x}^8 &= \xi^8 v^2 \\
\dot{x}^9 &= v^7 \\
\dot{\xi}^3 &= v^1 \\
\dot{\xi}^5 &= v^3 \\
\dot{\xi}^6 &= \zeta^6 v^2 \\
\dot{\zeta}^6 &= v^4 \\
\dot{\xi}^7 &= v^5 \\
\dot{\xi}^8 &= \zeta^8 v^2 \\
\dot{\zeta}^8 &= v^6
\end{aligned}\right. \tag{6.14}$$

We study the case where $a_1 = a_{2r+3} = \ldots = a_n = 0$. Following the section (6.2), we may assume that, in this case and for $r = 3$, the ideal I is spanned by:

$$\alpha_1 := dx^2 - x^3\,dx^4 - x^5\,dx^6 - x^7\,dx^8 \tag{6.15}$$

$$\alpha_2 := dx^3 - a_4\,dx^4 - a_5\,dx^5 - a_6\,dx^6 - a_7\,dx^7 - a_8\,dx^8 \tag{6.16}$$

The vector fields of the distribution that vanishes this forms make up the following system:

$$\left\{\begin{aligned}
\dot{x}^1 &= u^1 \\
\dot{x}^2 &= x^3u^2 + x^5u^4 + x^7u^6 \\
\dot{x}^3 &= \sum_{k=4}^{8} a_k u^{k-2} \\
\dot{x}^4 &= u^2 \\
\dot{x}^5 &= u^3 \\
\dot{x}^6 &= u^4 \\
\dot{x}^7 &= u^5 \\
\dot{x}^8 &= u^6 \\
\dot{x}^9 &= u^7
\end{aligned}\right. \tag{6.17}$$

Now prolong I by adding the forms:

$$\begin{aligned}
\alpha_{1,k} &= dx^{2k+1} - \xi^{2k+1}\,dx^4 \\
\alpha_{2,k} &= dx^{2k+2} - \xi^{2k+2}\,dx^4 \\
\alpha_{3,k} &= d\xi^{2k+2} - \zeta^{2k+2}\,dx^4
\end{aligned} \tag{6.18}$$

for $k = 2, 3$.

The new ideal is generated by $J := I + \{\alpha_{1,2}, \alpha_{2,2}, \alpha_{3,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_{3,3}\}$. Written in a more usual way:

$$
\begin{aligned}
u^1 &= v^6 \\
u^2 &= v^2 \\
u^3 &= \xi^5 v^2 \\
u^4 &= \xi^6 v^2 \\
u^5 &= \xi^7 v^2 \\
u^6 &= \xi^8 v^2 \\
u^7 &= v^7
\end{aligned}
$$

$$
\begin{aligned}
\dot{\xi}^6 &= \zeta^6 v^2 \\
\dot{\xi}^8 &= \zeta^8 v^2 \\
\dot{\xi}^5 &= v^1 \\
\dot{\xi}^7 &= v^3 \\
\dot{\zeta}^6 &= v^4 \\
\dot{\zeta}^8 &= v^5
\end{aligned}
$$

The prolongued system J is spanned by:

$$
\begin{aligned}
\alpha_1 &= dx^2 - \left(x^3 + x^5\xi^6 + x^7\xi^8\right)\, dx^4 \\
\alpha_2 &= dx^3 - \left(a_4 + \sum_{k=5}^{8} a_k \xi^k\right)
\end{aligned}
$$

and $\alpha_{1,2}, \alpha_{2,2}, \alpha_{3,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_{3,3}$.

Changing coordinates by

$$
\begin{aligned}
\widetilde{x}^1 &= a_4 + \sum_{k=5}^{8} a_k \xi^k + \sum_{l=2}^{3} \left(\xi^{2l+1}\xi^{2l+2} + x^{2l+1}\zeta^{2l+2}\right) \\
\widetilde{x}^3 &= x^3 + \sum_{k=2}^{3} x^{2k+1}\xi^{2k+2}
\end{aligned}
$$

leaving the other variables unchanged and noticing that

$$
d\widetilde{x}^3 \equiv dx^3 + \sum_{k=2}^{8} \left(\xi^{2k+1}\xi^{2k+2} + x^{2k+1}\zeta^{2k+2}\right)\, dx^4, \quad \mod (6.18)
$$

we see that J is spanned by

$$\begin{aligned}\overline{\alpha}_1 &= dx^2 - \widetilde{x}^3\, dx^4 \\ \overline{\alpha}_2 &= d\widetilde{x}^3 - \widetilde{x}^1\, dx^4\end{aligned}$$

and (6.18).

The forms belonging on J vanish the vector fields of the codistribution of the sytem that follows:

$$\begin{cases} \dot{x}^1 = v^6 \\ \dot{x}^2 = (x^3 + x^5\xi^6 + x^7\xi^8)v^2 \\ \dot{x}^3 = (a_4 + a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + a_8\xi^8)v^2 \\ \dot{x}^4 = v^2 \\ \dot{x}^5 = \xi^5 v^2 \\ \dot{x}^6 = \xi^6 v^2 \\ \dot{x}^7 = \xi^7 v^2 \\ \dot{x}^8 = \xi^8 v^2 \\ \dot{x}^9 = v^7 \\ \dot{\xi}^5 = v^1 \\ \dot{\xi}^6 = \zeta^6 v^2 \\ \dot{\xi}^7 = v^3 \\ \dot{\xi}^8 = \zeta^8 v^2 \\ \dot{\zeta}^6 = v^4 \\ \dot{\zeta}^8 = v^5 \end{cases} \tag{6.19}$$

6.4 Systems with 3 inputs and 5 states

In the previous sections we have dealed with systems of dimension 2 in $m+2$ variables. It has been proved that any controllable driftless system is flat. This result is also true for systems of dimension 2 in 5 variables but it is proved in a different way since lemma (6.2.1) cannot be applied.

Consider a driftless system of 2 inputs and 5 states

$$\dot{x} = \sum_{i=1}^{3} u_i f_i(x), \quad x \in \mathbb{R}^5$$

If I is such a Pfaffian system, the rank structre of its derived flag splits into four cases:

- $\dim I^{(1)} = 2$
- $\dim I^{(1)} = 1$, $\dim I^{(2)} = 1$
- $\dim I^{(1)} = 1$, $\dim I^{(2)} = 0$
- $\dim I^{(1)} = 0$

Clearly, the first two cases imply that the system is not controllable. Let's see some results coming from the last two cases.

6.4.1 Theorem *Let I be a Pfaffian system of dimension 2 and x_0 be a weakly regular point. Assume $\dim I^{(1)} = 1$ and $\dim I^{(2)} = 0$. Then,*

$$I = \{ dx^1 - x^2\, dx^4,\, dx^2 - x^3\, dx^4 \}$$

in suitable coordinates defined around x_0.

We will show that the Pfaffian system can be reduced to a unique normal form. To do this, we will state the following useful lemma.

6.4.2 Lemma *Let I be a totally nonhomologic Pfaffian system of dimension 2 in 5 variables and x_0 be a weakly regular point. Then I contains a form ω such that, around x_0,*

(i) $d\omega \wedge d\omega \wedge \omega = 0$

(ii) $d\omega \wedge \omega \neq 0$

A form ω holding this lemma is said to be rank 1 at x_0.

Proof: The proof is a particular case of lemma (6.2.1).

6.4.3 Theorem *Let I be a totally nonhomologic Pfaffian system of dimension 2 in 5 variables and x_0 be a weakly regular point. Then, in suitable coordinates defined around x_0, I takes one of the two following normal forms:*

- *If $\dim I^{(1)} = 1$ and $\dim I^{(2)} = 0$ then $I = \{ dx_1 - x_2\, dx_4,\, dx_2 - x_3\, dx_4 \}$.*
- *If $\dim I^{(1)} = 0$ then $I = \{ dx_1 - a(x)\, dx_3 - x_5\, dx_4,\, dx_2 - x_3\, dx_4 \}$ where a is some function.*

In both cases, I can be prolonged around x_0 into

$$J := \{ dz_1 - z_5\, dz_4,\, dz_2 - z_3\, dz_4,\, dz_3 - z_6\, dz_4 \}$$

Proof:
First of all, we will show that I can be prolonged into J.

- In the first case we have $I = \{ dx_1 - x_2\, dx_4,\, dx_2 - x_3\, dx_4 \}$. We prolong I by adding $\omega_3 = dx^5 - x^6\, dx_4$ and by performing the coordinate change:

$$\begin{aligned} z_1 &= x_5 \\ z_2 &= x_1 \\ z_3 &= x_2 \\ z_4 &= x_4 \\ z_5 &= x_6 \\ z_6 &= x_3 \end{aligned}$$

Clearly,

$$J = \{\, dz_2 - z_3\, dz_4,\, dz_3 - z_6\, dz_4,\, dz_1 - z_5\, dz_4\}$$

as desired.

- In the second case we have $I = \{\, dx_1 - a(x)\, dx_3 - x_5\, dx_4,\, dx_2 - x_3\, dx_4\}$. We prolong I by adding $\omega_3 = dx^3 - x_6\, dx_4$ and by performing the coordinate change:

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= x_3 \\ z_4 &= x_4 \\ z_5 &= x_5 + x_6 a(x) \\ z_6 &= x_6 \end{aligned}$$

Therefore, J is spanned by:

$$J = \{\, dz_1 - a(z)\, dz_3 - (z_5 - z_6 a(z))\, dz_4,\, dz_3 - z_6\, dz_4,\, dz_1 - z_5\, dz_4\}$$

The first form can be written as

$$\alpha_1 = dz_1 - a(z)\, dz_3 - (z_5 - z_6 a(z))\, dz_4 \equiv dz_1 - z_5\, dz_4 + z_6 a(z)\, dz_4$$

Choosing z_6 small:

$$J = \{\, dz_1 - z_5\, dz_4,\, dz_3 - z_6\, dz_4,\, dz_1 - z_5\, dz_4\}$$

as desired.

Now, we are going to show the two possible normal forms. By lemma (8.1.3), in this case, I contains a form of rank 1 at x_0. Up to a coordinate change I is spanned by:

$$\begin{aligned} \omega_1 &= \alpha\, dx_1 + \beta\, dx_3 + \gamma\, dx_4 + \delta\, dx_5 \\ \omega_2 &= dx_2 - x_3\, dx_4 \end{aligned}$$

where α, β, γ and δ are functions. Since $\dim I^0_{x_0} = 2$, one of this functions has to be different from zero at x_0. Therefore, we distinguish different cases:

- $\alpha(x_0) = \delta(x_0) = 0$

  We assume $\beta(x_0) \neq 0$, and without loss of generality $\beta(x) = -1$ around x_0. Notice that if $\beta(x_0) = 0$ and $\gamma(x_0) = 0$ with a coordinate change we exchange the roles of β and γ. We calculate the exterior derivative of ω_2:

$$d\omega_2 = dx_4 \wedge dx_3 = dx_4 \wedge (\alpha\, dx_1 + \gamma\, dx_4 + \delta\, dx_5 - \omega_1) \equiv \alpha\, dx_4 \wedge dx_1 + \delta\, dx_4 \wedge dx_5 \mod I$$

At the point x_0, $d\omega_2 \equiv 0 \mod$ which implies that $\dim I^{(1)}_{x_0} \geq 1$. But $\dim I^{(1)}_x \neq 2$ because the system has to be controllable. Furthermore, $\dim I^{(1)}_x$ must have constant dimension around x_0; thus α and δ equal 0 in a neighborhood of x_0. We conclude that, around x_0, $\dim I^{(1)}_x = 1$ and $\dim I^{(2)}_x = 0$. We have the hypothesis of theorem (8.1.2) so applying this theorem

$$I = \{\, dx_1 - x_2\, dx_4,\, dx_2 - x_3\, dx_4\}$$

then, the Pfaffian system is in the first normal form.

- $\alpha(x_0) \neq 0$

  In this case the system is spanned by

$$\begin{aligned} \omega_1 &= dx_1 - \overline{\beta}\, dx_3 - \overline{\gamma}\, dx_4 - \overline{\delta}\, dx_5 \\ \omega_2 &= dx_2 - x_3\, dx_4 \end{aligned}$$

  Notice that we can proceed as in section (6.2.2) by performing the coordinate change

$$\begin{aligned} z_1 &= g(x) \\ z_i &= x_i, \quad i = 2, \ldots, 5 \end{aligned}$$

  We get the following expressions of the generators of I:

$$\begin{aligned} \overline{\omega}_1 &= dz_1 - a(z)\, dz_3 - b(z)\, dz_4 \\ \overline{\omega}_2 &= dz_2 - z_3\, dz_4 \end{aligned}$$

  where a and b are functions. Computing the exterior derivative, we get

$$\begin{aligned} d\overline{\omega}_1 &\equiv c(z)\, dz_3 \wedge dz_4 + a_5(z)\, dz_3 \wedge dz_5 + b_5(z)\, dz_4 \wedge dz_5 \mod I \\ d\overline{\omega}_2 &= -\, dz_3 \wedge dz_4 \end{aligned}$$

  for a function $c(z)$ depending on $a(z)$, $b(z)$ and their partial derivatives. We distinguish two cases:

  - Case when $a_5(z_0) = b_5(z_0) = 0$

    This case implies that $I^{(1)}_{x_0} = 1$. But $\dim I^{(1)}_x$ must have constant dimension 1 around z_0, thus a_5 and b_5 equal 0 in a neighborhood of z_0 i.e the system does not depend on z_5. We apply the theorem (8.1.2) and we get

$$I = \{\, dz_1 - z_2\, dz_4,\, dz_2 - z_3\, dz_4\}$$

- Case when $b_5(z_0) \neq 0$
  Without loss of generality $b(z) = z_5$ around z_0. If $b_5(z_0) = 0$ and $a_5(z_0) \neq 0$ we can exchange the roles of a and b. This shows that I is spanned by

$$\begin{aligned} \overline{\omega}_1 &= dz_1 - a(z)\, dz_3 - z_5\, dz_4 \\ \overline{\omega}_2 &= dz_2 - z_3\, dz_4 \end{aligned}$$

as we wanted to show. ■

6.4.4 Corollary *Let*

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) \tag{6.20}$$

be a driftless system with 3 *inputs and* 5 *states. The following three statements are equivalent:*

1) (6.20) is 0*-flat*

2) (6.20) is flat

3) (6.20) is controllable.

if (6.20) satisfies one of these conditions then, around any weakly regular point, it can be put by dynamic feedback and coordinate change into the multi-input chained form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= z_3 v_1 \\ \dot{z}_3 &= v_2 \\ \dot{z}_4 &= z_5 v_1 \\ \dot{z}_5 &= z_6 v_1 \\ \dot{z}_6 &= v_3 \end{aligned}$$

Proof:

Clearly $(i) \Longrightarrow (ii) \Longrightarrow (iii)$. We only have to prove $(iii) \Longrightarrow (i)$.

Suppose (6.20) is controllable and x_0 is a weakly regular point. By theorem (6.4.3), the Pfaffian system $I = \{f_1, f_2, f_3\}^{\perp}$ can be assumed in a normal form around x_0.

We distinguish two cases

- $\dim I^{(1)} = 0$

We know that up to an invertible static feedback and a coordinate change, the system (6.20) can be written as

$$\begin{aligned}
\dot{x}_1 &= a(x)u_1 + x_5u^2 \\
\dot{x}_2 &= x_3u_2 \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= u_2 \\
\dot{x}_5 &= u_3
\end{aligned}$$

Clearly, $y_1 = x_1, y_2 = x_2, y_3 = x_4$ are flat outputs of the system. Let's see this

$$\begin{aligned}
x_1 &= y_1 \\
x_2 &= y_2 \\
x_3 &= \frac{\dot{y}_2}{\dot{y}_3} \\
x_4 &= y_3 \\
u_1 &= \frac{\ddot{y}_2\dot{y}_3 - \ddot{y}_3\dot{y}_2}{(\dot{y}_3)^2} \\
u_2 &= \dot{y}_3
\end{aligned}$$

The variables x_5 is obtained by solving the following equation

$$\dot{y}_1 = a\left(y_1, y_2, \frac{\dot{y}_2}{\dot{y}_3}, y_2, x_5\right)\frac{\ddot{y}_2\dot{y}_3 - \ddot{y}_3\dot{y}_2}{(\dot{y}_3)^2} + x_5\dot{y}_3$$

from here we find the expression of u_3 by differentiation. Thus, the system is flat at every point $(x_0, \overline{u}_0)$ such that x_0 is weakly regular $u_0^2 \neq 0$ and $a_5(x_0)u_0^1 + u_0^2 \neq 0$. These points form a dense open subset of $X \times \overline{U}$.

Apply now the dynamic feedback

$$\dot{x}_6 = v_3, \quad u_1 = x_6v_1, \quad u_2 = v_1$$
$$u_3 = \frac{v_2 - [(x_5 + x_6a(x))a_1(x) + x_3a_2(x) + x_6a_3(x) + a_4(x)]x_6v_1 - a(x)v_3}{1 + x_6a_5(x)}$$

and the coordinate change

$$\begin{aligned}
z_1 &= x_4 \\
z_2 &= x_1 \\
z_3 &= x_5 + x_6a(x) \\
z_4 &= x_2 \\
z_5 &= x_3 \\
z_6 &= x_6
\end{aligned}$$

which are well-defined as soon as x_6 is small enough, to put the system into multi-input chained form.

- $\dim I^{(1)} = 1$ and $\dim I^{(2)} = 0$ the system reads

$$\begin{aligned}
\dot{x}_1 &= x_2 u_2 \\
\dot{x}_2 &= x_3 u_2 \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= u_2 \\
\dot{x}_5 &= u_3
\end{aligned}$$

Now, $y_1 = x_1, y_2 = x_2, y_3 = x_5$ are flat outputs. Let's see this

$$\begin{aligned}
x_1 &= y_1 \\
x_2 &= \frac{\dot{y}_1}{\dot{y}_2} \\
x_3 &= \frac{\ddot{y}_1 \dot{y}_2 - \ddot{y}_2 \dot{y}_1}{(\dot{y}_2)^3} \\
x_4 &= y_2 \\
x_5 &= y_3 \\
u_1 &= \left(\frac{\ddot{y}_1 \dot{y}_2 - \ddot{y}_2 \dot{y}_1}{(\dot{y}_2)^3} \right)' \\
u_2 &= \dot{y}_2 \\
u_3 &= \dot{y}_3
\end{aligned}$$

■

6.5 Application to an spherical robot

In this section we consider an spherical rolling robot actuated by internal rotors as described in [12]. Under a proper placement of the rotors the center of mass of the composite system is at the geometric center of the sphere and, as a result, the gravity does not enter the motion equations.

To describe the system, we introduce the coordinate frames shown at figure 6.1. There, Σ_b is an inertial frame fixed at the base, Σ_o is a frame fixed at the geometric center of the sphere, Σ_a is a frame fixed at the contact plane. In addition, at the contact point we introduce the contact frame of the object Σ_{co}, and the contact frame of the plane, Σ_{ca}.

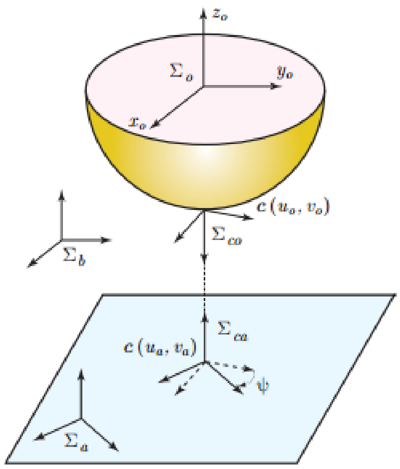

Figure 6.1: System formalization

The contact coordinates are given by the angles u_o and v_o, describing the contact point on the sphere, and ua and va, describing the contact point on the plane, and by the contact angle ψ which is defined as the angle between the x-axis of Σ_{co} and Σ_{ca}. In what follows, we will assume that the frame Σ_a coincides with Σ_b, and the frame Σ_{ca} is parallel to Σ_a. We will also assume that in the initial configuration $u_a = 0$, $v_a = 0$, $u_o = 0$, $v_o = 0$, $\psi = 0$. Therefore $\Sigma_{ca}(0) = \Sigma_b$ and in the initial configuration the axis of Σ_o are parallel to those of Σ_b.

Let $\omega_{\mathbf{o}} = \{\omega_x, \omega_y, \omega_z\}^T$ be the angular velocity of the frame Σ_o, defined in projections onto the axis of the base frame Σ_b. Let R be the radius of the sphere. Then, the evolution of the contact coordinates is described by the following system

$$\begin{pmatrix} \dot{u}_a \\ \dot{v}_a \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 & R & 0 \\ -R & 0 & 0 \\ -\sin\psi/\cos v_o & -\cos\psi/\cos v_o & 0 \\ -\cos\psi & \sin\psi & 0 \\ -\sin\psi\tan v_o & -\cos\psi\tan v_o & -1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{6.21}$$

Our goal is to show that this driftless system is controllable and can be put into multi-input chained form by dynamic feedback and coordinate change.

For simplicity we define this new variables

$$\begin{aligned} x_1 &= u_a \\ x_2 &= v_a \\ x_3 &= u_o \\ x_4 &= v_o \\ x_5 &= \psi \end{aligned}$$

The vector fields of the system are

$$g_1 = \begin{pmatrix} 0 \\ -R\cos x_4 \\ -\sin x_5 \\ -cos x_4 \cos x_5 \\ -\sin x_4 \sin x_5 \end{pmatrix}, \quad g_2 = \begin{pmatrix} R\cos x_4 \\ 0 \\ -\cos x_5 \\ \cos x_4 \sin x_5 \\ -sin x_4 \cos x_5 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{6.22}$$

We have to found two 1-forms such that the contraction with all the vector fields to be zero. This forms could be, for example:

$$\begin{aligned} \alpha_1 &= \cos x_5\, dx_1 - \sin x_5\, dx_2 + R\cos x_4\, dx_3 \\ \alpha_2 &= \sin x_5\, dx_1 + \cos x_5\, dx_2 - R\, dx_4 \end{aligned}$$

Its exterior derivative are

$$\begin{aligned} d\alpha_1 &= \cos x_5\, dx_5 \wedge dx_1 - \sin x_5\, dx_5\ w\, dx_2 \\ d\alpha_2 &= -\sin x_5\, dx_5 \wedge dx_1 - \cos x_5\, dx_5 \wedge dx_2 + R\sin x_4\, dx_4 \wedge dx_3 \end{aligned}$$

Consider

$$\begin{aligned} A &= a_1\, dx_1 + b_1\, dx_2 + c_1\, dx_3 + d_1\, dx_4 + e_1\, dx_5 \\ B &= a_2\, dx_1 + b_2\, dx_2 + c_2\, dx_3 + d_2\, dx_4 + e_2\, dx_5 \end{aligned}$$

Defining $I = \{\alpha_1, \alpha_2\}$, we will find $I^{(1)}$:

$$\begin{aligned} d\alpha_2 = A \wedge \alpha_1 + B \wedge \alpha_2 = &-a_1 \sin x_5\, dx_1 \wedge dx_2 + a_1 R\cos x_4\, dx_1 \wedge dx_3 \\ &+ b_1 \cos x_5\, dx_2 \wedge dx_1 + Rb_1 \cos x_4\, dx_2 \wedge dx_3 + c_1 \cos x_5\, dx_3 \wedge dx_1 \\ &- c_1 \sin x_5\, dx_3 \wedge dx_2 + d_1 \cos x_5\, dx_4 \wedge dx_1 - d_1 \sin x_5\, dx_4 \wedge dx_2 \\ &+ d_1 R\cos x_4\, dx_4 \wedge dx_3 + e_1 \cos x_5\, dx_5 \wedge dx_1 - e_1 \sin x_5\, dx_5 \wedge dx_2 \\ &+ e_1 R\cos x_4\, dx_5 \wedge dx_3 + a_2 \cos x_5\, dx_1 \wedge dx_2 - Ra_2\, dx_1 \wedge dx_4 \\ &+ b_2 \sin x_5\, dx_2 \wedge dx_1 - Rb_2\, dx_2 \wedge dx_4 + c_2 \sin x_5\, dx_3 \wedge dx_1 \\ &+ c_2 \cos x_5\, dx_3 \wedge dx_2 - Rc_2\, dx_3 \wedge dx_4 + d_2 \sin x_5\, dx_4 \wedge dx_1 \\ &+ d_2 \cos x_5\, dx_4 \wedge dx_2 + e_2 \sin x_5\, dx_5 \wedge dx_1 + e_2 \cos x_5\, dx_5 \wedge dx_2 \\ &- e_2 R\, dx_5 \wedge dx_4 \end{aligned}$$

This expression leads to a several constrains. Three of them are:

$$\begin{aligned} \cos x_5 &= e_1 \cos x_5 + e_2 \sin x_5 \\ -\sin x_5 &= -e_1 \sin x_5 + e_2 \cos x_5 \\ e_1 R \cos x_4 &= 0 \end{aligned}$$

which is not possible since $e_1 = 1$ and $e_2 = 0$. Similarly, for $d\alpha_1$ one finds that

$$\begin{aligned} -\sin x_5 &= e_1 \cos x_5 + e_2 \sin x_5 \\ -\cos x_5 &= -e_1 \sin x_5 + e_2 \cos x_5 \\ e_2 R &= 0 \end{aligned}$$

which is not possible. Then,

$$\begin{aligned} \dim I^{(0)} &= 2 \\ \dim I^{(1)} &= 0 \end{aligned}$$

Therefore, by theorem (6.4.3), I can be expressed in the following manner:

$$I = \{\, dx_1 - a(x)\, dx_3 - x_5\, dx_4,\, dx_2 - x_3\, dx_4\}$$

for some function $a(x)$. Let's show this. Consider the second generator, α_2. We calculate its rank.

$$\begin{aligned} d\alpha_2 \wedge \alpha_2 &= (\cos x_5\, dx_5 \wedge dx_1 + \sin x_5\, dx_2 \wedge dx_5) \wedge (\sin x_5\, dx_1 + \cos x_5\, dx_2 - R\, dx_4) = \\ &= dx_1 \wedge dx_2 \wedge dx_5 - R \cos x_5\, dx_1 \wedge dx_4 \wedge dx_5 + R \sin x_5\, dx_2 \wedge dx_4 \wedge dx_5 \neq 0 \\ d\alpha_2 \wedge \alpha_2 &= 0 \end{aligned}$$

Then, the rank of α_2 is $r = 1$. We apply Pfaff's theorem to write α_2 into a normal form. A function $f_1 = a_1\, dx_1 + \cdots + a_5\, dx_5$ such that $d\alpha_2 \wedge \alpha_2 \wedge df_1 = 0$ has to be found.

$$\begin{aligned} d\alpha_2 \wedge \alpha_2 \wedge df_1 = &-a_3\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 - a_4\, dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 - \\ &- R \cos x_5 a_2\, dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 - R \cos x_5 a_3\, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 - \\ &- R \sin x_5 a_1\, dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + R \sin x_5 a_3\, dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \end{aligned}$$

We can choose, for example, $f_1 = x_5$. Now, since $r = 1$, we are looking for a function f_2 such that

$$\begin{aligned} \alpha_2 \wedge df_1 \wedge df_2 &= 0 \\ df_1 \wedge df_2 &\neq 0 \end{aligned}$$

Setting $df_2 = g_1\, dx_1 + g_2\, dx_2 + g_3\, dx_3 + g_4\, dx_4 + g_5\, dx_5$ we do the computations:

$$
\begin{aligned}
\alpha_2 \wedge df_1 \wedge df_2 = {} & (\sin x_5\, dx_1 + \cos x_5\, dx_2 - R\, dx_4) \wedge dx_5 \wedge (g_1\, dx_1 + g_2\, dx_2 + g_3\, dx_3 + g_4\, dx_4 + g_5\, dx_5) = \\
= {} & -g_2 \sin x_5\, dx_1 \wedge dx_2 \wedge dx_5 - g_3 \sin x_5\, dx_1 \wedge dx_3 \wedge dx_5 - g_4 \sin x_5\, dx_1 \wedge dx_4 \wedge dx_5 + \\
& + g_1 \cos x_5\, dx_1 \wedge dx_2 \wedge dx_5 - g_3 \cos x_5\, dx_2 \wedge dx_3 \wedge dx_5 - g_4 \cos x_5\, dx_2 \wedge dx_4 \wedge dx_5 - \\
& - Rg_1\, dx_1 \wedge dx_4 \wedge dx_5 - Rg_2\, dx_2 \wedge dx_4 \wedge dx_5 - Rg_3\, dx_3 \wedge dx_4 \wedge dx_5 = 0
\end{aligned}
$$

This leads the following constrains:

$$
\begin{aligned}
g_3 &= 0 \\
-g_2 \sin x_5 + g_1 \cos x_5 &= 0 \\
-Rg_2 - g_4 \cos x_5 &= 0 \\
-Rg_1 - g_4 \sin x_5 &= 0
\end{aligned}
$$

We choose, for instance

$$
\begin{aligned}
g_1 &= \sin x_5 \\
g_2 &= \cos x_5 \\
g_4 &= -R
\end{aligned}
$$

Then,

$$
df_2 = \sin x_5\, dx_1 + \cos x_5\, dx_2 - R\, dx_4 + g_5\, dx_5
$$

Therefore,

$$
f_2 = x_1 \sin x_5 + x_2 \cos x_5 - Rx_4
$$

and we determine g_5:

$$
g_5 = x_1 \cos x_5 - x_2 \sin x_5
$$

Once we have found f_1 and f_2, α_2 can be written in a normal form as follows

$$
\alpha_2 = df_2 - g\, df_1 = dz_2 - z_3\, dz_4
$$

for some function g that will be determine now. It is clear that

$$
\begin{aligned}
z_2 &= x_1 \sin x_5 + x_2 \cos x_5 - Rx_4 \\
z_4 &= x_5
\end{aligned}
$$

Therefore,

$$z_3 = x_1 \cos x_5 - x_2 \sin x_5$$

Now, we want

$$\alpha_1 = \alpha\, dz_1 + \beta\, dz_3 + \gamma\, dz_4 + \delta\, dz_5$$

On the other hand

$$\alpha_1 = \cos x_5\, dx_1 - \sin x_5\, dx_2 + R \cos x_4\, dx_3$$

Imposing the following values, α_1 is written as desired.

$$\begin{aligned}
\alpha &= R \cos x_4 \\
\beta &= 1 \\
\gamma &= x_1 \sin x_5 + x_2 \cos x_5 \\
\delta &= 0 \\
z_1 &= x_3 \\
z_5 &= \frac{x_1 \sin x_5 + x_2 \cos x_5}{R \cos x_4}
\end{aligned}$$

with $x_4 \neq k\pi/2, k \in \mathbb{Z}$. Then,

$$\begin{aligned}
\alpha_2 &= dz_2 - z_3\, dz_4 \\
\alpha_1 &= \alpha\, dz_1 + dz_3 + \gamma\, dz_4
\end{aligned}$$

In fact,

$$\begin{aligned}
z_1 &= x_3 \\
z_2 &= x_1 \sin x_5 + x_2 \cos x_5 - R x_4 \\
z_3 &= x_1 \cos x_5 - x_2 \sin x_5 \\
z_4 &= x_5 \\
z_5 &= \frac{x_1 \sin x_5 + x_2 \cos x_5}{R \cos x_4}
\end{aligned}$$

with $x_4 \neq k\pi/2, k \in \mathbb{Z}$, is a diffeomorphism. We are now in position to apply the constructive demonstration of theorem (6.4.3).

Since $\alpha \neq 0$, we can divide the 1-form by this value:

$$\overline{\alpha}_1 = dz_1 - \frac{1}{R \cos x_4}\, dz_3 - \frac{x_1 \sin x_5 + x_2 \cos x_5}{R \cos x_4}\, dz_4$$

Setting $a(x) = \frac{1}{R\cos x_4}$,

$$\overline{\alpha}_1 = dz_1 - \overline{a}(z)\, dz_3 - z_5\, dz_4$$

where $\overline{a}(z)$ is $a(x)$ expressed in the new variables.

Then, the Pfaffian system is generated by

$$I = \{\, dz_1 - \overline{a}(z)\, dz_3 - z_5\, dz_4,\, dz_2 - z_3\, dz_4\}$$

In order to fin the flat outputs of the system, we prolong I with

$$\alpha_3 = dz_3 - z_6\, dz_4$$

and we perform the following change of variables:

$$\begin{aligned}
\bar{z}_1 &= z_1 \\
\bar{z}_2 &= z_2 \\
\bar{z}_3 &= z_3 \\
\bar{z}_4 &= z_4 \\
\bar{z}_5 &= z_5 + z_6\overline{a}(z) \\
\bar{z}_6 &= z_6
\end{aligned}$$

Therefore the generators of I become

$$\begin{aligned}
\alpha_1 &= dz_1 - \bar{a}(z)\, dz_3 - z_5\, dz_4 = dz_1 - (\bar{a}(z)z_6 + z_5)\, dz_4 = d\bar{z}_1 - \bar{z}_5\, d\bar{z}_4 \\
\alpha_2 &= dz_2 - z_3\, dz_4 = d\bar{z}_2 - \bar{z}_3\, d\bar{z}_4 \\
\alpha_3 &= dz_3 - z_6\, dz_4 = d\bar{z}_3 - \bar{z}_6\, d\bar{z}_4
\end{aligned}$$

If we want to express the system in the state variables we have to seek for 3 vector fields

$$g_i = (g_i^1, g_i^2, g_i^3, g_i^4, g_i^5, g_i^6), \quad i = 1, 2, 3$$

such that

$$g_i \lrcorner \alpha_j = 0, \quad i, j = 1, 2, 3$$

in other words

$$\begin{aligned} g_i^1 - \bar{z}_3 g_i^4 &= 0 \\ g_i^2 - \bar{z}_3 g_i^4 &= 0 \\ g_i^3 - \bar{z}_6 g_i^4 &= 0 \end{aligned}$$

for $i = 1, 2, 3$. One solution could be

$$\begin{aligned} g_1 &= (\bar{z}_5, \bar{z}_3, \bar{z}_6, 1, 0, 0) \\ g_2 &= (0, 0, 0, 0, 1, 0) \\ g_3 &= (0, 0, 0, 0, 0, 1) \end{aligned}$$

Therefore, the system reads

$$\begin{cases} \dot{\bar{z}}_1 = \bar{z}_5 v_1 \\ \dot{\bar{z}}_2 = \bar{z}_3 v_1 \\ \dot{\bar{z}}_3 = \bar{z}_6 v_1 \\ \dot{\bar{z}}_4 = v_1 \\ \dot{\bar{z}}_5 = v_2 \\ \dot{\bar{z}}_6 = v_3 \end{cases}$$

Thus, the flat outputs of the system are

$$\begin{aligned} y_1 &= \bar{z}_1 \\ y_2 &= \bar{z}_2 \\ y_3 &= \bar{z}_4 \end{aligned}$$

In fact, all the state variables are expressed in terms of flat outputs and its derivatives. Let's see this:

$$\begin{aligned} v_1 &= \dot{y}_3 \\ v_2 &= \frac{\ddot{y}_1 \dot{y}_3 - \dot{y}_1 \ddot{y}_3}{\dot{y}_3^3} \\ \bar{z}_1 &= y_1 \\ \bar{z}_2 &= y_2 \\ \bar{z}_3 &= \frac{\dot{y}_2}{\dot{y}_3} \\ \bar{z}_4 &= y_3 \\ \bar{z}_5 &= \frac{\dot{y}_1}{\dot{y}_3} \\ \bar{z}_6 &= \frac{\ddot{y}_2 \dot{y}_3 - \dot{y}_2 \ddot{y}_3}{\dot{y}_3^3} \end{aligned}$$

In the original state variables the flat outputs are expressed as follows:

$$
\begin{aligned}
y_1 &= x_3 \\
y_2 &= x_1 \sin x_5 + x_2 \cos x_5 - R x_4 \\
y_3 &= x_5
\end{aligned}
$$

7

The Gardner-Shadwick algorithm for exact linearization to Brunovsky normal form

We present an algorithm using the minimal number of integrations for the exact linearization of nonlinear systems to Brunovsky normal form under feedback.

In this section we will define the Brunovsky normal form and we will explain how to compute the explicit formula for the feedback of a controllable linearizable system.

To achieve this goal we will make use of the classical Frobenius theorem (3.3.4).

7.1 The Brunovsky normal form

Let us consider a control system with n states and m inputs given by

$$\frac{dx}{dt} = F(x,u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{7.1}$$

This structure defines a Pfaffian system

$$\begin{aligned} dx^1 - F^1(x,u)\,dt &= 0 \\ dx^2 - F^2(x,u)\,dt &= 0 \\ &\vdots \\ dx^n - F^n(x,u)\,dt &= 0 \end{aligned}$$

We will see that if the system satisfies certain congruences, then the diffeomorphisms on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ preserving the Pfaffian system are of the form

$$\Phi(t,x,u) = (t, \phi(x), \psi(x,u))$$

7.1.1 Definition A control system is in a *Brunovsky normal form* if there exists integers $k_1 \geq \cdots \geq k_m \geq 0$ (*Kronecker indexes*) and independent functions $t, x^1_1, \ldots, x^1_{k_1}, x^2_1, \ldots, x^2_{k_2}, \ldots, x^m_1, \ldots, x^m_{k_m}, u_1, \ldots, u_m$ such that its associated Pfaffian system I has generators of the form

$$
\begin{array}{llll}
\omega_1^1 = dx_1^1 - x_2^1\,dt, & \omega_2^1 = dx_2^1 - x_3^1\,dt, & \cdots, & \omega_{k_1}^1 = dx_{k_1}^1 - u_1\,dt \\
\omega_1^2 = dx_1^2 - x_2^2\,dt, & \omega_2^2 = dx_2^2 - x_3^2\,dt, & \cdots, & \omega_{k_2}^2 = dx_{k_2}^2 - u_2\,dt \\
 & \vdots & & \\
\omega_1^m = dx_1^m - x_2^m\,dt, & \omega_2^m = dx_2^m - x_3^m\,dt, & \cdots, & \omega_{k_m}^m = dx_{k_m}^m - u_m\,dt
\end{array}
\tag{7.2}
$$

To analyze the derived flag of the system (7.2) we calculate the exterior derivatives of the generators. Hence, for $1 \leq i \leq m$ and $1 \leq j \leq k_i - 1$ we obtain

$$
d\omega_j^i = dt \wedge dx_{j+1}^i = dt \wedge (\,dx_{j+1}^i - x_{j+2}^i\,dt) = dt \wedge \omega_{j+1}^i \tag{7.3}
$$

and for $j = k_i$:

$$
d\omega_{k_i}^i = dt \wedge du_i
$$

The equations (7.3) enables us to decompose the system I into m towers using the successive derived flags and the Kronecker indexes.

Consider the subsystem $I_j \subset I$. This ideal is generated by

$$
I_j = \{\omega_1^j, \omega_2^j, \ldots, \omega_{k_j}^j\}
$$

Recall that

$$
I_j^{(1)} = \{\alpha \in I_j : d\alpha \equiv 0 \mod I_j\}
$$

Since $d\omega_{k_j}^j = dt \wedge du_j$, the previous ideal is spanned by

$$
I_j^{(1)} = \{\omega_1^j, \omega_2^j, \ldots, \omega_{k_j-1}^j\}
$$

The second derived flag is defined as

$$
I_j^{(2)} = \{\alpha \in I_j^{(1)} : d\alpha \equiv 0 \mod I_j^{(1)}\}
$$

Since $d\omega_{k_j-1}^j = dt \wedge d\omega_{k_j}^j$ and $\omega_{k_j}^j \notin I_j^{(1)}$, the above ideal is spanned by

$$
I_j^{(2)} = \{\omega_1^j, \omega_2^j, \ldots, \omega_{k_j-2}^j\}
$$

Proceeding by induction we find the structure of the j-th tower:

$$
\begin{array}{lll}
\omega_1^j & \omega_2^j & \cdots \omega_{k_j}^j \\
\vdots & & \\
\omega_1^j & \omega_2^j & \\
\omega_1^j & &
\end{array}
$$

This tower has k_j rows that corresponds to the derived length of I_j. Now, if we collect all the m towers together taking into account that $k_1 \geq k_2 \geq \cdots \geq k_m > 0$ we obtain

$$\begin{array}{cccccccccc} \omega_1^j & \omega_2^j & \cdots & \omega_{k_j}^j & \cdots & \cdots & \omega_1^m & \omega_2^m & \cdots & \omega_{k_m}^m \\ \vdots & & & & & & \vdots & & & \\ \vdots & & & & & & \omega_1^m & & & \\ \omega_1^j & \omega_2^j & & & & & & & & \\ \omega_1^j & & & & & & & & & \end{array} \tag{7.4}$$

Notice that in the first row we find the generators of I. The succeeding rows taken together generate the derived flag of I.

We define

$$d_j = \dim I^{(j)}/I^{(j+1)} = \text{number of towers with at least } j+1 \text{ rows}$$

It is clear that k_j is the number of rows at the j-th tower, hence k_{d_j} is the number of rows in the d_j-th tower. Defining $D = \{d_1, \cdots, d_{k_1-1}\}$ then we can understant d_j and the larger integer belonging in D such that $k_{d_j} \geq j+1$.

The derived length of I is k_1 and for each j, k_j is the number of rows in the j-th tower. These numbers are determined by the integers d_j which implies that the Kronecker indexes are not only feedback invariants but also invariants for the full diffeomorphism group on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$.

Although the derived systems $I^{(j)}$ are diffeomorphism invariant, their bases are not, and a change of a basis has the effect of replacing the equations (7.3) by congruences modulo the appropriate derived systems. This leads to the following congruences modulo $I^{(j)}$ for $1 \leq j \leq k_j$.

$$\left\{ \begin{array}{ll} d\omega^1_{k_1-j} & \equiv dt \wedge \omega^1_{k_1-j+1} \\ & \vdots \\ d\omega^m_{k_{d_j}-j} & \equiv dt \wedge \omega^m_{k_{d_j}-j+1} \end{array} \right. , \quad \text{mod } I^{(j+1)} \tag{7.5}$$

Any system which is diffeomorphic to a Brunovsky normal form has generatoros which satisfy these congruences. The next theorem shows that the converse is also true that is if the Pfaffian system has generators satisfying the congruences (7.5) then the control system can be put into a Brunovsky normal form. This proof yields the Gardner - Shadwick algorithm for the exact linearization problem in a form with minimal integrations.

7.1.2 Theorem *A control system with distinct Kronecker indexes satisfying $k_1 > k_2 > \ldots > k_m > 0$ can be put into Brunovsky normal form by a nonlinear feedback transformation if and only if there is a set of generators for its associated Pfaffian system satisfying the congruences (7.5).*

Proof:
We have shown that the derived length of I is k_1, therefore $I^{(k_1-1)}$ has only one generator called ω_1^1. By definition, whose coefficients are independent of time. Consider the funcions b_1^1 and η_1^1, that are independent of time. We define the gerenator as:

$$\omega_1^1 = \eta_1^1 - b_1^1\, dt$$

By hypothesis, this generator satisfies the congruences (7.5), thus

$$d\omega_1^1 \equiv dt \wedge d\omega_2^1 \mod \omega_1^1$$

So it is true that

$$d\omega_1^1 \wedge \omega_1^1 \wedge dt = 0$$

Using the definition of ω_1^1 the previous relation implies

$$(\, d\eta_1^1 - \, db_1^1 \wedge \, dt) \wedge (\eta_1^1 - b_1^1\, dt) \wedge \, dt = 0 \Longrightarrow \, d\eta_1^1 \wedge \eta_1^1 \wedge \, dt = 0$$

Since η_1^1 is independent of time we obtain $d\eta_1^1 \wedge \eta_1^1 = 0$.

As a result, η_1^1 is a completely integrable 1-form on the state space and any first integral y_1^1 of $\eta_1^1 = 0$ satisfies

$$\eta_1^1 = \mu_1^1\, dy_1^1 \Longrightarrow \, dy_1^1 = \frac{\eta_1^1}{\mu_1^1}$$

where μ_1^1 is independent of time. We choose a new generator $\overline{\omega}_1^1$ defined as

$$\overline{\omega}_1^1 = \frac{1}{\mu_1^1}\omega_1^1 \tag{7.6}$$

we define a new function $y_2^1 := \frac{b_1^1}{\mu_1^1}$. Therefore, the new generator becomes

$$\overline{\omega}_1^1 = \frac{\eta_1^1}{\mu_1^1} - \frac{b_1^1}{\mu_1^1}\, dt = \, dy_1^1 - y_2^1\, dt \tag{7.7}$$

The functions y_1^1 and y_2^1 are functionally independent:

$$-\, dy_2^1 \wedge \, dt \wedge \, dy_1^1 = \, d\overline{\omega}_1^1 \wedge \overline{\omega}_1^1 = \frac{1}{(\mu_1^1)^2}\, d\omega_1^1 \wedge \omega_1^1 \neq 0$$

By construction, they are state-space coordinates. We now have

$$d\overline{\omega}_1^1 = -\, dy_2^1 \wedge \, dt = \, dt \wedge \, dy_2^1 \tag{7.8}$$

Using (7.6):

$$d\overline{\omega}_1^1 = \frac{1}{\mu_1^1}\, d\omega_1^1 - \frac{d\mu_1^1 \wedge \omega_1^1}{(\mu_1^1)^2} \equiv \frac{1}{\mu_1^1} \mod \omega_1^1$$

Or in other words, using the hypothesis:

$$d\overline{\omega}_1^1 = \frac{1}{\mu_1^1}\, dt \wedge \omega_2^1 \mod \overline{\omega}_1^1$$

Substracting (7.8) from the previous relation we obtain:

$$dt \wedge \left(\frac{1}{\mu_1^1}\omega_2^1 - dy_2^1 \right) + \alpha\overline{\omega}_1^1 = 0$$

Doing the wedge product with $\overline{\omega}_1^1$:

$$dt \wedge \left(\frac{1}{\mu_1^1}\omega_2^1 - dy_2^1 \right) \wedge \overline{\omega}_1^1 = dt \wedge \left(\frac{1}{\mu_1^1}\omega_2^1 - dy_2^1 \right) \wedge dy_1^1 = 0 \tag{7.9}$$

Next, we write

$$\frac{1}{\mu_1^1}\omega_2^1 = \eta_2^1 - b_2^1\, dt \tag{7.10}$$

with η_2^1 and b_2^1 functions independent of time. Then, (7.9) gives

$$\left(\eta_2^1 - dy_2^1\right) \wedge dy_1^1 \wedge dt = 0$$

hence

$$\left(\eta_2^1 - dy_2^1\right) \wedge dy_1^1 = 0$$

and we deduce that

$$\eta_2^1 - dy_2^1 = c_2^1\, dy_1^1 \tag{7.11}$$

with c_2^1 a function independent of time. Now using (7.7), (8.2.2) and (7.11):

$$\begin{aligned} \frac{1}{\mu_1^1}\omega_2^1 - dy_2^1 &= \eta_2^1 - b_2^1\, dt - dy_2^1 = -b_2^1\, dt + c_2^1\, dy_1^1 = \\ &= -b_2^1\, dt + c_2^1(\, dy_1^1 - y_2^1\, dt) + c_2^1 y_2^1\, dt = \\ &= c_2^1\overline{\omega}_1^1 - (b_2^1 - c_2^1 y_2^1)\, dt \end{aligned}$$

Reordering the terms:

$$\frac{1}{\mu_1^1}\omega_2^1 - c_2^1\overline{\omega}_1^1 = dy_2^1 - (b_2^1 - c_2^1 y_2^1)\, dt$$

Setting

$$\overline{\omega}_2^1 = \frac{1}{\mu_1^1}\omega_2^1 - c_2^1\overline{\omega}_1^1$$

and

$$y_3^1 := b_2^1 - c_2^1 y_2^1$$

We find the expression for the new generator:

$$\overline{\omega}_2^1 = dy_2^1 - y_3^1\, dt$$

It is true that

$$-\, dy_3^1 \wedge dt \wedge dy_2^1 \wedge dy_1^1 = d\overline{\omega}_2^1 \wedge \overline{\omega}_2^1 \wedge \overline{\omega}_1^1 \neq 0$$

so y_1^1, y_2^1 and y_3^1 are functionally independent and by construction are state-space coordinates. In particular we see that

$$d\overline{\omega}_1^1 = dt \wedge \overline{\omega}_2^1$$

which replaces the congruence (7.5) by equality and allows us to proceed by induction.

This algorithm proceeds inductively until we reach $\omega_{k_1-k_2+1}$ at which time a congruence from the second tower appears. At this stage, the first tower up to the level $k_1 - k_2 + 1$ is in a Brunovsky normal form. Where the last mnodified generator is $\overline{\omega}_{k_1-k_2+1}$:

$$\overline{\omega}_{k_1-k_2+1} \equiv \frac{1}{\mu^1_{k_1-k_2+1}}\omega^1_{k_1-k_2+1} \quad \mathrm{mod}\ \left(\overline{\omega}_1^1, \overline{\omega}_2^1, \ldots, \overline{\omega}^1_{k_1-k_2}\right)$$

and this generator is defined as:

$$\overline{\omega}^1_{k_1-k_2+1} = dy^1_{k_1-k_2+1} - y^1_{k_1-k_2+1}\, dt$$

as above, we can show that $y_1^1, y_2^1, \ldots, y^1_{k_1-k_2+1}$ are functionally independents of time and have been found as the coefficients of dt in the modified generators.

Recall that by hypothesis we have the condition

$$d\omega^1_{k_1-k_2+1} \equiv dt \wedge \omega^1_{k_1-k_2+1} \quad \mathrm{mod}\ \left(\overline{\omega}_1^1, \ldots, \overline{\omega}^1_{k_1-k_2}, \omega^1_{k_1-k_2+1}, \omega_1^2\right)$$

where ω_1^2 is the new generator that appears at the second column and by hypothesis satisfies

$$d\omega_1^2 \equiv dt \wedge \omega_2^2 \quad \mathrm{mod}\ \left(\overline{\omega}_1^1, \ldots, \overline{\omega}^1_{k_1-k_2}, \omega^1_{k_1-k_2+1}, \omega_1^2\right)$$

Now

$$d\overline{\omega}^1_{k_1-k_2+1} \equiv \frac{1}{\mu_{k_1-k_2+1}\, d\omega^1_{k_1-k_2+1}} \equiv dt \wedge \left(\frac{1}{\mu^1_{k_1-k_2}}\right)\omega^1_{k_1-k_2+2} \mod \left(\overline{\omega}^1_1, \ldots, \overline{\omega}^1_{k_1-k_2}, \omega^1_{k_1-k_2+1}, \omega^2_1\right)$$

and letting

$$\overline{\omega}_{k_1-k_2+2} = \left(\frac{1}{\omega_{k_1-k_2}}\right)\omega_{k_1-k_2+2}$$

we have

$$\left\{ \begin{array}{l} d\overline{\omega}^1_{k_1-k_2+1} \equiv dt \wedge \overline{\omega}^1_{k_1-k_2+2} \\ d\omega^2_1 \equiv dt \wedge \omega^2_2 \end{array} \right. , \quad \mod \left(\overline{\omega}^1_1, \ldots, \overline{\omega}^1_{k_1-k_2}, \overline{\omega}^1_{k_1-k_2+1}, \omega^2_1\right)$$

Now let

$$\omega^2_1 = \eta^2_1 - b^2_1\, dt$$

where η^2_1 and b^2_1 are functions independent of time. We know that

$$d\omega^2_1 \wedge \omega^2_1 \wedge \overline{\omega}^1_1 \wedge \cdots \wedge \overline{\omega}^1_{k_1-k_2+1} \wedge dt = 0$$

which implies that

$$d\eta^2_1 \wedge \eta^2_1 \wedge dy^1_1 \wedge \cdots \wedge dy^1_{k_1-k_2+1} \wedge dt = 0$$

And since all the previous function are independent of time

$$d\eta^2_1 \wedge \eta^2_1 \wedge dy^1_1 \wedge \cdots \wedge dy^1_{k_1-k_2+1} = 0$$

We can apply the corollary of Frobenius theorem (3.3.4) that says that there exists functions $\mu^2_1, y^2_1, a^1_1, \ldots, a^1_{k_1-k_2+1}$ which are independent of time such that

$$\eta^2_1 = \mu^2_1\, dy^2_1 + a^1_1\, dy^1_1 + \cdots + a^1_{k_1-k_2+1}\, dy_{k_1-k_2+1}$$

As a result

$$\frac{1}{\mu^2_1}\left(\eta^2_1 - a^1_1\overline{\omega}^1_1 - \cdots - a^1_{k_1-k_2+1}\overline{\omega}^1_{k_1-k_2+1}\right) = dy^2_1 + \frac{1}{\mu^2_1}\left(a^1_1 y^1_2 + \cdots + a^1_{k_1-k_2+1} y^1_{k_1-k_2+1}\right) dt$$

We set the modified generator as

$$\begin{aligned}\overline{\omega}_1^2 &= \frac{1}{\mu_1^2}\left(\omega_1^2 - a_1^1\overline{\omega}_1^1 - \cdots - a^1_{k_1-k_2+1}\overline{\omega}^1_{k_1-k_2+1}\right)\\ &= dy_1^2 + \frac{1}{\mu_1^2}\left(-b_1^2\,dt + a_1^1 y_2^1 + \ldots + a^1_{k_1-k_2+1}y^1_{k_1-k_2+1}\right)\,dt\end{aligned}$$

And we define y_2^2 as the coefficient of dt in $\overline{\omega}_1^2$. Thus:

$$y_2^2 = b_1^2\,dt - a_1^1 y_2^1 - \ldots - a^1_{k_1-k_2+1}y^1_{k_1-k_2+1}$$

And we have

$$\overline{\omega}_1^2 = dy_1^2 - y_2^2\,dt$$

This procedure is now inductive through the second tower until the level $k_1 - k_3 + 1$. At this stage we proceed as above.

Notice that with this procedure the new controls are defined depending on the old controls and state variables. Thus we have defined a non-linear feedback transformation.

The previous theorem does not cover the case of multiplicity of the Kronecker indexes. Suppose

$$\begin{aligned}\nu_1 &= \text{multiplicity of } k_1\\ &\vdots\\ \nu_l &= \text{multiplicity of } k_{\nu_1+\cdots+\nu_{l-1}+1}\end{aligned}$$

As a consequence of the definitions we have that

$$\sum_{i=1}^{l}\nu_i = p$$

Notice that in the case where all the multiplicities are 1 then $l = p$. With this definitions we introduce the following theorem:

7.1.3 Theorem (Exact linearization with minimal integration) *A control system can be put into Brunovsky normal form by a nonlinear feedback transformation if and only if there is a set of generators for its associated Pfaffian system satisfying the congruences (7.5).*

The minimum number of integrations needed are l decoupled completely integrable systems of dimensions $\nu_1, \ldots, \nu_l$.

Proof:
The case where all the multiplicities are 1 is treated in theorem (8.1.2). Thus we need to treat the procedure when a Kronecker index has multiplicity greater than one. Let us assume that k_q is the first Kronecker index with multiplicity greater than 1. Then we have

$$\nu_1 = 1, \cdots, \nu_{q-1} = 1, \nu_q > 1$$

We apply the theorem (8.1.2) up to the stage q, that is when the first basis element of the tower q appears. Then, all the forms up to this level have been modified. The generators for $I^{(q)}$ are:

$$I^{(q)} = \left\{ \overline{\omega}_1^1, \ldots, \overline{\omega}_q^1, \ldots, \overline{\omega}_1^{q-1}, \ldots, \overline{\omega}_q^{q-1}, \omega_1^q, \ldots, \omega_1^{q+\nu_q+1} \right\}$$

By theorem (8.1.2) and the hypothesis we have the following equations and congruences:

$$d\overline{\omega}_\alpha^1 = dt \wedge \overline{\omega}_{\alpha+1}^i, \quad 1 \leq i \leq q-1, 1 \leq \alpha \leq q$$

and

$$\begin{array}{rcl} d\omega_1^q & \equiv & dt \wedge \omega_2^q \\ & \vdots & \\ d\omega_1^{q+\nu_q-1} & \equiv & dt \wedge \omega_2^{q+\nu_q-1} \end{array} , \quad \text{mod } I^{(q)} \tag{7.12}$$

In a similar way as in theorem (8.1.2), we consider

$$\omega_1^j = \eta_1^j - b_1^j \, dt, \quad q \leq j \leq q + \nu_q - 1$$

where η_1^j and b_1^j are independent functions of time. Next, we observe that

$$d\omega_1^j \wedge \omega_1^q \wedge \ldots \wedge \omega_1^{q+\nu_q-1} \wedge \overline{\omega}_1^1 \wedge \ldots \wedge \overline{\omega}_q^{q-1} \wedge dt = 0$$

which implies

$$d\eta_1^j \wedge \eta_1^q \ldots \wedge \eta_1^{q+\nu_q-1} \wedge dy_1^1 \wedge \ldots \wedge dy_q^{q-1} \wedge dt = 0$$

By independence,

$$\eta_1^q, \ldots, \eta_q^{q+\nu_q-1}, dy_1^1, \ldots, y_q^{q-1}$$

is a q-dimensional completely integrable system on the state-space. By the corollary of the Frobenius theorem, one can find a system $\{y_1^q, \ldots, y_1^{q+\nu_q-1}\}$ of first integrals independent of $\{y_1^1, \ldots, y_q^{q-1}\}$. This may be achieved by setting $y_1^1, \ldots, y_q^{q-1}$ equal to constants and integrating the resulting ν_q-dimensional Frobenius system. This implies that there is a nonsingular matrix of state-space functions A such that

$$\begin{pmatrix} \eta_1^q \\ \vdots \\ \eta_1^{q+\nu_q-1} \end{pmatrix} = A \begin{pmatrix} dy_1^q \\ \vdots \\ dy_1^{q+\nu_q-1} \end{pmatrix} := A \, dY_1^q \tag{7.13}$$

As a result

$$\Omega_1^q := \begin{pmatrix} \omega_1^q \\ \vdots \\ \omega_1^{q+\nu_q-1} \end{pmatrix} = A\, dY_1^q - B_1^q\, dt \tag{7.14}$$

where

$$B_1 := \begin{pmatrix} b_1^q \\ \vdots \\ b_1^{q+\nu_q-1} \end{pmatrix}$$

Now introducing a vector of one-form

$$\overline{\Omega}_1^q = A^{-1}\Omega_1^q = dY_1^q - A^{-1}B_1^q\, dt$$

and defining a vector of function

$$Y_2^q := \begin{pmatrix} y_2^q \\ \vdots \\ y_2^{q+\nu_q-1} \end{pmatrix}$$

by

$$Y_2^q = A^{-1}B_1^q$$

we see that

$$\overline{\Omega}_1^q = dY_1^q - Y_2^q\, dt$$

and

$$d\overline{\Omega}_1^q = -\, dY_2^q \wedge dt = dt \wedge dY_2^q$$

Since

$$d\overline{\Omega}_1^q = A^{-1}\, d\Omega_1^q + d(A^{-1})\Omega_1^q \equiv A^{-1}\, d\Omega_1^q \mod I^{(q)}$$

and using (7.12) we get

$$d\overline{\Omega}_1^q \equiv A^{-1}\, d\Omega_1^q \equiv dt \wedge A^{-1}\Omega_2^q \mod I^{(q)}$$

where

$$\Omega_2^q = \begin{pmatrix} \omega_2^1 \\ \vdots \\ \omega_2^{q+\nu_q-1} \end{pmatrix}$$

substracting the two previous relations we get

$$dt \wedge (\, dY_2^q - A^{-1}\Omega_2^q) \equiv 0, \quad \mod I^{(q)}$$

which implies

$$dY_2^q - A^{-1}\Omega_2^q = C_2^q\Theta_2^q + Y_3^q\,dt$$

where Θ_2^q is the basis

$$\Theta_2^q = \left(\overline{\omega}_1^1, \ldots, \overline{\omega}_q^1, \ldots, \overline{\omega}_1^{q-1}, \ldots, \overline{\omega}_q^{q-1}, \overline{\omega}_1^q, \ldots, \overline{\omega}_1^{q+\nu_q+1}\right)^t$$

of $I^{(q)}$, and C_2^q, Y_3^q are independent of time. Letting

$$\Omega_2^q = A^{-1}\Omega_2^q + C_2^q\Theta_2^q = dY_2^q - Y_3^q\,dt$$

From here we can proceed inductively. ■

The constructive proofs of the two theorems addressed in this section allow us to construct an algorithm with 5 steps presented in the following section.

7.2 The description of the GS algorithm

We divide this algorithm into 5 steps as follows:

- Step 1: Compute the successive derived flags for I.
- Step 2: Build the m-towered structure determining the congruences (7.5).
- Step 3: Modify the generators ω_j^i to obtain $\overline{\omega}_j^i$. This generators will replace the congruences by equalities.
- Step 4: Use the Frobenius theorem to integrate the systems needed to put the generators of the towers into a normal form.
- Step 5: Set the coefficients of dt in the modified generators as the new state variables.

7.2.1 Example Consider the Hunt-Su-Meyer system described in [6]:

$$\begin{aligned}
\dot{x}_1 &= \sin x_2 \\
\dot{x}_2 &= \sin x_3 \\
\dot{x}_3 &= (x_4)^3 + u_1 \\
\dot{x}_4 &= x_5 + (x_4)^3 - (x_1)^{10} \\
\dot{x}_5 &= u_2
\end{aligned}$$

with 5 state variables and 2 inputs. As we have seen, this structure leads to a Pfaffian system I spanned by:

$$
\begin{aligned}
\omega_1 &= dx_1 - \sin x_2\, dt \\
\omega_2 &= dx_2 - \sin x_3\, dt \\
\omega_3 &= dx_3 - \left((x_4)^3 + u_1\right)\, dt \\
\omega_4 &= dx_4 - \left(x_5 + (x_4)^3 - (x_1)^{10}\right)\, dt \\
\omega_5 &= dx_5 - u_2\, dt
\end{aligned}
$$

The first step is to calculate the exterior derivative to construct the successive derived flags:

$$
\begin{aligned}
d\omega_1 &= -\cos x_2\, dx_2 \wedge\, dt \\
d\omega_2 &= -\cos x_3\, dx_3 \wedge\, dt \\
d\omega_3 &= -3(x_4)^2\, dx_4 \wedge\, dt -\, du_1 \wedge\, dt \\
d\omega_4 &= 10(x_1)^9\, dx_1 \wedge\, dt - 3(x_4)^2\, dx_4 \wedge\, dt -\, dx_5 \wedge\, dt \\
d\omega_5 &= -\, du_2 \wedge\, dt
\end{aligned}
$$

Then $I^{(0)} = \langle \omega_1, \omega_2, \cdots \omega_5 \rangle$. The first derived flag is

$$
I^{(1)} = \{\alpha \in I^{(0)} \mid d\alpha \equiv 0 \mod I^{(0)}\}
$$

Therefore, since

$$
\begin{aligned}
d\omega_1 &= -\cos x_2\, dx_2 \wedge\, dt = \cos x_2\, dt \wedge \omega_2 \\
d\omega_2 &= -\cos x_3\, dx_3 \wedge\, dt = \cos x_3\, dt \wedge \omega_3 \\
d\omega_4 &= 10(x_1)^9\, dx_1 \wedge\, dt - 3(x_4)^2\, dx_4 \wedge\, dt -\, dx_5 \wedge\, dt = -10(x_1)^9\, dt \wedge \omega_1 + 3(x_4)^2\, dt \wedge \omega_4 +\, dt \wedge \omega_5
\end{aligned}
$$

the deriveg flag is $I^{(1)} = \langle \omega_1, \omega_2, \omega_4 \rangle$.

Proceeding in a similar way, we find $I^{(2)} = \langle \omega_1 \rangle$ and $I^{(3)} = 0$ therefore $k_1 = 3$. Since $k_1 + k_2 = 5$, the other Kronecker index is $k_2 = 2$.

Now the second step is to construct the two towers as follows:

$$
\begin{array}{ccccc}
\omega_1^1 & \omega_2^1 & \omega_3^1 & \omega_1^2 & \omega_2^2 \\
\omega_1^1 & \omega_2^1 & & \omega_1^2 & \\
\omega_1^1 & & & &
\end{array}
\tag{7.15}
$$

We set $\omega_1^1 = \omega_1$ and calculate its exterior derivative:

$$
d\omega_1^1 = \, dt \wedge \cos x_2 \omega_2
$$

Defining $\omega_2^1 = \cos x_2 \omega_2$ the previous relations becomes

$$
d\omega_1^1 = \, dt \wedge \omega_2^1
$$

Notice that the second row of the first tower is filled out.

Now we are at the stage $k_1 - k_2 + 1 = 2$ so a new generator has appeared at the second tower that is $\omega_1^2 = \omega_4$. We calculate the exterior derivative of ω_2^1:

$$\begin{aligned} d\omega_2^1 &= -\sin x^2\, dx^2 \wedge \omega^2 + \cos x^2\, d\omega^2 = \sin x^2 \sin x^3\, dx^2 \wedge\, dt - \cos x^2 \cos x^3\, dx^3 \wedge\, dt = \\ &= dt \wedge (-\sin x^2 \sin x^3\, dx^2 + \cos x^2 \cos x^3\, dx^3) \equiv \\ &\equiv dt \wedge (\cos x^2 \cos x^3 \omega^3) \mod I^{(1)} \end{aligned}$$

We set $\omega_3^1 = \cos x^2 \cos x^3 \omega^3$, hence

$$d\omega_2^1 \equiv dt \wedge \omega_3^1 \mod I^{(1)}$$

We do the same with ω_1^2:

$$d\omega_1^2 = -10(x_1)^9\, dt \wedge \omega^1 + 3(x_4)^2\, dt \wedge \omega^4 +\, dt \wedge \omega^5 \equiv\, dt \wedge \omega^5 \mod I^{(1)}$$

We set $\omega_2^2 = \omega^5$.

The following step is to modify the generators in order to achieve the equalities instead of equivalences in the exteior derivatives.

$$d\omega_1^1 =\, dt \wedge \omega_2^1$$

is already an equality. We do not have to modify ω_2^1. For $d\omega_2^1$:

$$\begin{aligned} d\omega_2^1 &=\, dt \wedge (-\sin x_2 \sin x_3 \omega^2 + \cos x_2 \cos x_3 \omega^3) = \\ &=\, dt \wedge \left(\frac{-\sin x_2 \sin x_3}{\cos x_2} \omega_2^1 + \omega_3^1 \right) \end{aligned}$$

We set

$$\overline{\omega}_3^1 = \frac{-\sin x_2 \sin x_3}{\cos x_2} \omega_2^1 + \omega_3^1$$

which yields

$$d\omega_2^1 =\, dt \wedge \overline{\omega}_3^1$$

Finally, for $d\omega_1^2$:

$$d\omega_1^2 =\, dt \wedge \left(-10(x_1)^9 \omega_1^1 + 3(x_4)^2 \omega_1^2 + \omega_2^2 \right)$$

Defining

$$\overline{\omega}_2^2 = -10(x_1)^9\omega_1^1 + 3(x_4)^2\omega_1^2 + \omega_2^2$$

The previous expression becomes

$$d\omega_1^2 = dt \wedge d\overline{\omega}_2^2$$

The last two steps are based on the integration of the systems to put the basis of the towers in a normal form and the obtention of the linearizing controls. Recall, the first generator is

$$\omega_1^1 = dx_1 - \sin x_2\, dt$$

We consider $y_1^1 = x_1$ and $y_2^1 = \sin x_2$. Thus,

$$\omega_1^1 = dy_1^1 - y_2^1\, dt$$

that is written in a normal form. Since $dy_2^1 = \cos x_2\, dx_2$, ω_2^1 can be written as

$$\omega_2^1 = \cos x_2\, dx_2 - \sin x_3 \cos x_2\, dt = dy_2^1 - y_3^1\, dt$$

where $y_3^1 = \sin x_3 \cos x_2$. The exterior derivative of y_3^1 is

$$dy_3^1 = \cos x_2 \cos x_3\, dx_3 - \sin x_2 \sin x_3\, dx_2$$

The generator $\overline{\omega}_3^1$ becomes:

$$\begin{aligned}
\overline{\omega}_3^1 &= \frac{-\sin x_2 \sin x_3}{\cos x_2}\omega_2^1 + \omega_3^1 = \\
&= -\sin x_2 \sin x_3\, dx_2 + \sin x_2 \sin^2 x_3\, dt + \cos x_2 \cos x_3\, dx_3 - \cos x_2 \cos x_3 \left((x_4)^3 + u_1\right) dt = \\
&= dy_3^1 - v_1\, dt
\end{aligned}$$

where we have found the first linearizing control:

$$v_1 = -\sin x_2 \sin^2 x_3 + \cos x_2 \cos x_3 \left((x_4)^3 + u_1\right)$$

For ω_1^2:

$$\omega_1^2 = dx_4 - (x_5 + (x_4)^3 - (x_1)^{10})\, dt$$

Considering $y_1^2 = x_4$ and $y_2^2 = x_5 + (x_4)^3 - (x_1)^{10}$ we can put ω_1^2 in a normal form

$$\omega_1^2 = dy_1^2 - y_2^2\, dt$$

Since

$$dy_2^2 = dx_5 + 3(x_4)^2\, dx_4 - 10(x_1)^9\, dx_1$$

then proceeding as before, $\overline{\omega}_2^2 = dy_2^2 - v_2\, dt$ where

$$v_2 = u^2 + 3(x_4)^2(x_5 + (x_4)^3 - (x_1)^{10}) - 10(x_1)^9 \sin(x_2)$$

that is the second linearizing control.

Since we have the following diffeomorphism between the original and the new states variables:

$$\begin{aligned}
y_1^1 &= x_1 \\
y_2^1 &= \sin x_2 \\
y_3^1 &= \sin x_3 \cos x_2 \\
y_1^2 &= x_4 \\
y_2^2 &= x_5 + (x_4)^3 - (x_1)^{10}
\end{aligned}$$

The linearized system reads

$$\begin{aligned}
\dot{y}_1^1 &= y_2^1 \\
\dot{y}_2^1 &= y_3^1 \\
\dot{y}_3^1 &= v_1 \\
\dot{y}_1^2 &= y_2^2 \\
\dot{y}_2^2 &= v_2
\end{aligned}$$

and the flat outputs are

$$\begin{aligned}
\alpha_1 &= y_1^1 \\
\alpha_2 &= y_1^2
\end{aligned}$$

and the feedbacks are given by

$$\begin{aligned}
v_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\ddot{\alpha}_1 \\
v_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\dot{\alpha}_2
\end{aligned}$$

To simulate the behaviour of the new variables we proceed as follows:

- We impose initial and final conditions to the original state variables $x = (x, y, \theta, \phi)$.

- We will obtain, trhough the diffeomorphism, the corresponding initial and final conditions for $y = (y_1^1, y_2^1, y_3^1, y_1^2, y_2^2)$ and for $\alpha = (\alpha_1, \dot{\alpha}_1, \ddot{\alpha}_1, \alpha_2, \dot{\alpha}_2)$ afterwards.
- We will construct two polynomials with the degree according to the conditions in order to express the flat outputs in terms of time.

$$P_5(t) = \alpha_1(t)$$
$$Q_3(t) = \alpha_2(t)$$

- Once we have found the polynomials we can express the inputs in terms of time by differentiating each polinomial:

$$v_1 = \frac{\mathrm{d}^3}{\mathrm{d}t^3} P_5(t)$$
$$v_2 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} Q_3(t)$$

The following conditions are imposed:

$$x(0) = (0, 0, 1, 3, \pi) \longrightarrow y(0) = (0, 0, 1/2, 3, 28)$$
$$x(1) = (-1, 1, 1, 2, 0) \longrightarrow y(1) = (-1, 0, 0, 2, 7)$$

We simulate the trajectories:

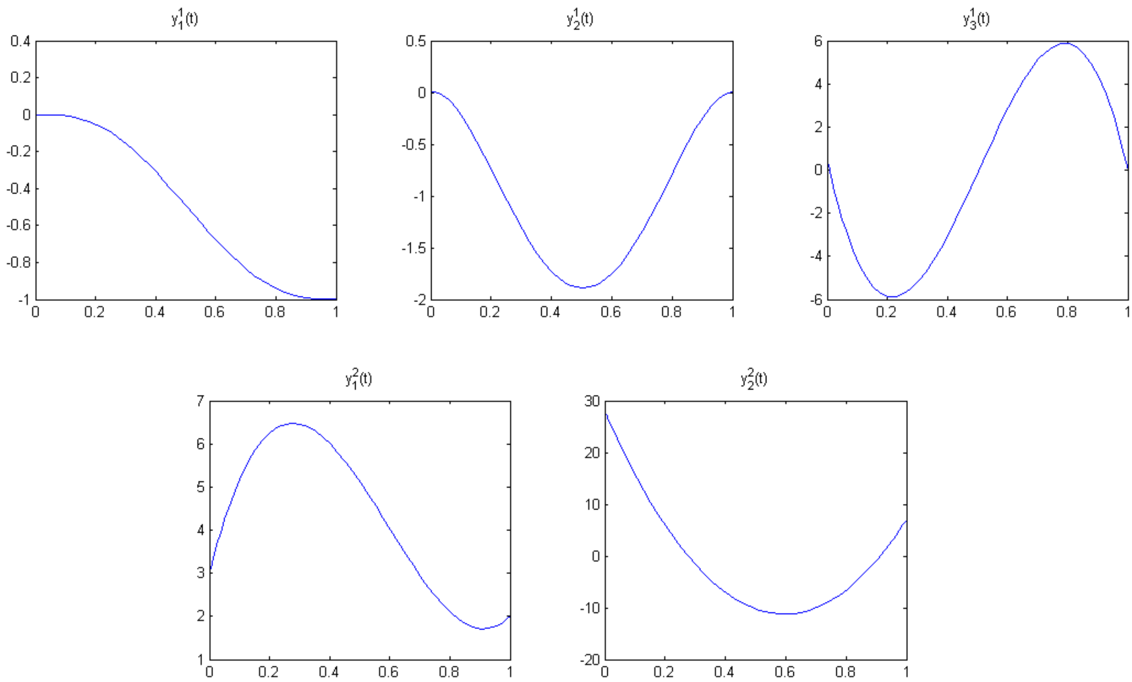

Figure 7.1: Behaviour of the $\{y\}$ variables

8
Feedback linearization in driftless systems

In this section we will study feedback linearization of driftless systems in terms of dynamic immersions. We will give, therefore, a few definitions and some results involving this concept. Afterwards, we will use this formulation to give a computable condition for feedback linearization in terms of Lie Brackets and we will prove that this condition is necessary when the number of inputs is equal to two.

8.1 Main results

8.1.1 Definition Let be $\dot{x} = f(x, u)$ a control system such that $f : X \times U \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$. This system is said to be *feedback linearizable* if we are able to find a smooth dynamic feedback

$$\begin{aligned} \dot{z} &= a(x, z, v) \\ u &= \sigma(x, z, v) \end{aligned}$$

where a, σ are defined over $\widetilde{X} \times Z \times V \subset X \times \mathbb{R}^r \times \mathbb{R}^q$ such that the system

$$\begin{aligned} \dot{x} &= f(x, \sigma(x, z, v)) \\ \dot{z} &= a(x, z, v) \end{aligned}$$

is diffeomorphic on $\widetilde{X} \times Z$ to a controllable linear system.

8.1.2 Theorem (Cartan) *Consider a Pfaffian system of dimension n, and let r be the dimension of the retracting space $C(I)$. Then, there are coordinates $(z_1, \ldots, z_r)$ such that*

$$I = \langle b_1^k(z)\, dz_1 + \cdots + b_r^k(z)\, dz_r, \quad k = 1, \ldots, s\rangle$$

This means that the generators of the Pfaffian system can be written depending on r variables instead of n.

8.1.3 Lemma *Let I be a Pfaffian system of dimension $s \geq 2$, such that dim $I^{(1)} = s - 1$ and dim $I^{(2)} = s - 2$. Then dim $C(I) = s + 2$ and dim $C\left(I^{(1)}\right) = s + 1$.*

Proof:
It is clear that under linear combinations, the derived flags become

$$
\begin{aligned}
I^{(0)} &= \{\alpha_1, \ldots, \alpha_s\} \\
I^{(1)} &= \{\alpha_1, \ldots, \alpha_{s-1}\} \\
I^{(2)} &= \{\alpha_1, \ldots, \alpha_{s-2}\}
\end{aligned}
$$

Thus, using the definition of derived flag we have that

$$
\begin{aligned}
d\alpha_k &\equiv 0 \mod I^{(1)}, \quad k = 1, \ldots, s-2 \\
d\alpha_{s-1} &\equiv 0 \mod I^{(0)} \\
d\alpha_{s-1} &\not\equiv 0 \mod I^{(1)}
\end{aligned}
$$

Hence, there exists a form α_{s+1} independent of $\alpha_1, \ldots, \alpha_s$ such that

$$
d\alpha_{s+1} \equiv \alpha_s \wedge \alpha_{s+1} \mod I^{(1)}
$$

We want to find the dimension of the retracting space $C(I^{(1)})$. To do that, we consider a vector field ξ such that $\xi \lrcorner \alpha_1 = \ldots = \xi \lrcorner \alpha_{s-1} = 0$, we have to impose $\xi \lrcorner\, d\alpha_k \in I^{(1)}, \quad k = 1, \ldots, s-1$, but:

$$
\begin{aligned}
&\xi \lrcorner\, d\alpha_k \equiv 0 \mod I^{(1)} \quad k = 1, \ldots, s-2 \\
&\xi \lrcorner\, d\alpha_{s-1} \equiv (\xi \lrcorner \alpha_s)\, \alpha_{s+1} - (\xi \lrcorner \alpha_{s+1})\, \alpha_s \mod I^{(1)}
\end{aligned}
\tag{8.1}
$$

From here we deduce that the retracting space is

$$
C(I^{(1)}) = \{\xi \in \mathscr{X}(M), \xi \lrcorner \alpha_1 = \ldots = \xi \lrcorner \alpha_{s+1} = 0\}^{\perp} = \{\alpha_1, \ldots, \alpha_{s+1}\}
$$

Thus, the dimension of $C(I^1)$ is $s+1$.

By construction $C(I^1)$ satisfies the Frobenius condition. In particular $d\alpha_s \equiv 0 \mod C(I^{(1)})$, but $d\alpha_s \not\equiv 0 \text{mod } I^0$ hence, there exists a form α_{s+2} independent of $\alpha_1, \ldots, \alpha_{s+1}$ such that

$$
d\alpha_s \equiv \alpha_{s+1} \wedge \alpha_{s+2} \mod I^0
$$

We proceed as before. We consider a vector field ξ such that $\xi \lrcorner \alpha_1 = \ldots = \xi \lrcorner \alpha_s = 0$ we impose the condition that remains to belong in the retracting space. We have

$$
d\alpha_s \lrcorner \xi \equiv (\xi \lrcorner \alpha_{s+1})\, \alpha_{s+2} - (\xi \lrcorner \alpha_{s+2})\, \alpha_{s+1} \mod I^{(0)}
$$

On the other hand, we find using (8.1) that

$$
\xi \lrcorner\, d\alpha_k \equiv 0 \mod I^{(0)} \quad k = 1, \ldots, s-2
$$

when $\xi\lrcorner\alpha_1 = \ldots = \xi\lrcorner\alpha_s = 0$, hence

$$C(I^0) = \{\xi \in \mathscr{X}(M), \xi\lrcorner\alpha_1 = \ldots = \xi\lrcorner\alpha_{s+2} = 0\}^\perp = \{\alpha_1, \ldots, \alpha_{s+2}\}$$

■

Before getting into the concept of dynamic inmersions we will state a simplified verion of the Pfaff's theorem:

8.1.4 Lemma *Let $I = \{\alpha\}$ be a Pfaffian system of dimension 1 such that $d\alpha \wedge \alpha \neq 0$. Then in suitable coordinates*

$$I = \{\, dz_1 + z_2\, dz_3 + \sum_{i=4}^{n} b_i(z)\, dz_i\}$$

The proof is deduced from Frobenius' theorem.

8.2 Dynamic inmersion

Refering to the definition of a control system as in (7.1), this system defines a Paffian system on $\mathbb{R} \times X \times U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

$$I_f = \{\, dx^i - f^i(x,u)\, dt, \quad i = 1, \ldots n\}$$

8.2.1 Definition Consider g defined on an open set $Y \times V \subset \mathbb{R}^p \times \mathbb{R}^q$. We say that $\dot{x} = f(x,u)$ is *equivalent by invertible static feedback* to $\dot{y} = g(y,v)$ at (x_0, u_0) if and only if there is a diffeomorphism ψ defined on $\widetilde{Y} \times \widetilde{V} \subset Y \times V$

$$\psi(t,y,v) = (t, \varphi(y), \kappa(y,v))$$

with $\varphi\left(\widetilde{Y}\right) \times \kappa\left(\widetilde{Y} \times \widetilde{V}\right)$ a neighborhood of (x_0, u_0), such that

$$\psi^*(I_f) = I_g$$

where

$$I_g = \{\, dy^i = g^i(y,v)\, dt, \quad i = 1, \ldots, p\}$$

If, moreover, $\kappa(y,v) = v$ we say that the systems are *conjugate*.

8.2.2 Definition The system $\dot{x} = f(x,u)$ is *dynamically inmersed* in $\dot{y} = g(y,v)$ at (x_0, u_0) if there exist a map κ from an open subset $\widetilde{Y} \times \widetilde{V} \subset Y \times V$ to a neighborhood of u_0 and a submersion φ from $\widetilde{Y}$ to a neighborhood of x_0 such that $\phi^* I_f \subset I_g$.

8.2.3 Definition If the previous definition is satisfied at every point (x_0, u_0) of a dense open subset $X \times U$, we say that $\dot{x} = f(x, u)$ is *dynamically inmsersed* in $\dot{y} = g(y, v)$.

8.2.4 Theorem *We say that $\dot{x} = f(x, u)$ is dynamically inmersed in $\dot{y} = g(y, v)$ at (x_0, u_0) if and only if there exists a dynamic feedback called B, defined as*

$$\begin{aligned} \dot{z} &= a(x, z, v) \\ u &= \sigma(x, z, v) \end{aligned}$$

defined around (x_0, z_0, v_0) with $u_0 = \sigma(x_0, z_0, v_0)$, such that the closed-loop system called f_B, given by

$$\begin{aligned} \dot{x} &= f(x, \sigma(x, z, v)) \\ \dot{z} &= a(x, z, v) \end{aligned}$$

is conjugate to $\dot{y} = g(y, v)$ at (x_0, z_0, v_0).

Once we have the condition to put a system dynamically inmersi into another one, we can give the following result.

8.2.5 Corollary *A system $\dot{x} = f(x, u)$ is feedback linearizable if and only if it is dynamically inmersed in a controllable linear system.*

Notice that we can assume that a linear controllable system $\dot{y} = g(y, v)$ defined on a subset of $\mathbb{R}^p \times \mathbb{R}^q$ is in a Brunosvky form. Recall that setting

$$\begin{aligned} y &:= \left(y_0^1, \ldots, y_{d_1-1}^1, \ldots, y_0^q, \ldots, y_{d_q-1}^q\right) \\ v &:= \left(y_{d_1}^1, \ldots, y_{d_q}^q\right) \end{aligned}$$

where $d_1 + \cdots + d_q = p$ we have q chains of d_i integrators

$$\omega_j^i = dy_{j-1}^i - y_j^i \, dt, \quad i = 1, \ldots, q, \quad j = 1, \ldots, d_i \tag{8.2}$$

The associated Pfaffian system is given by

$$C_{d_1, \ldots, d_q}^q := I_g = \{\omega_1^1, \ldots, \omega_{d_1}^1, \ldots, \omega_1^q, \ldots, \omega_{d_q}^q\}$$

8.3 Dynamic inmersions in driftless systems

We restrict us in this section to a driftless systems of the form

$$\dot{x} = \sum_{i=1}^{m} u^i f_i(x) \tag{8.3}$$

a system with m inputs and n states, where $f_1, \ldots, f_m$ are vector fields on X. We know that the associated Pfaffian system to (8.3) is given by

$$I = \{f_1, \ldots, f_m\}^{\perp}$$

Consider that I_f has generators $\alpha_1, \ldots, \alpha_{n-m}, \beta_1, \ldots, \beta_m$ of the form

$$\alpha_k = \sum_{i=1}^{m} a_k^i(x)\, dx^k, \quad k = 1, \ldots, n-m$$
$$\beta_k = \sum_{i=1}^{n} b_k^j(x)\, dx^k - u^j\, dt, \quad j = 1, \ldots, m$$

where u^j are the inputs of the system. Clearly, $I = I_f^{(1)}$.

One important result involving I is that does not depend neither on time nor inputs, i.e, is a Pfaffian system on $\widetilde{X} \subset X$. The interesting point is that I contains all we need to study the feedback linearization of the driftless system (8.3). To get an idea of the role of I_f, we will give a definition and a condition when the system (8.3) is feedback linearizable.

8.3.1 Definition A Pfaffian system I on X is *linearizable* at $x_0 \in X$ if there exists submersion φ from an open set $Y \subset \mathbb{R}^{q+d_1+\cdots+d_q}$ to a neighborhood of x_0 such that $\varphi^* I \subset C^q_{d_1,\ldots,d_q}$ for some positive integers $q, d_1, \ldots, d_q$.

8.3.2 Definition If the previous property holds at every point x_0 of a dense subset of X, we say that I is *linearizable.*

8.3.3 Proposition *A driftless system $\dot{x} = \sum_{i=1}^{m} u^i f_i(x)$ is feedback linearizable in the sense of (8.2.5) using (8.2.2) if and only if the Pfaffian system I is linearizable in the sense of (8.3.1).*

Proof:
$\Longrightarrow)$
If $\dot{x} = \sum_{i=1}^{m} u^i f_i(x)$ is feedback linearizable then, there exists a map

$$\psi(t, y, v) = (y, \phi(y), \kappa(y, v))$$

where ϕ is a submersion such that $\psi^* I_f \subset I_g = C^q_{d_1,\ldots,d_q}$. We want to show that $\varphi^* I \subset C^q_{d_1,\ldots,d_q}$.

Recall that I is a Pfaffian system on $\widetilde{X} \subset X$, therefore $\psi^* I = \phi^* I$. Since $I \subset I_f$, clearly $\psi^* I \subset \psi^* I_f$. By hypothesis, $\psi^* I_f \subset C_{d_1,\ldots,d_q}$. Thus, we have proved that $\varphi^* I \subset C^q_{d_1,\ldots,d_q}$.

$\Longleftarrow$)
Suppose that I is linearizable in the sense of (8.3.1) i.e, there exists a submersion φ such that $\varphi^* I \subset C^q_{d_1,\dots,d_q}$. Since $\varphi^* I = \psi^* I$, the map ψ will pull back the generators of I_f into the associated Pfaffian system of a controllable linear system, let's see this:

Consider I_f generated by $\alpha_1, \dots, \alpha_{n-m}, \beta_1, \dots, \beta_m$. Clearly $\psi^* \alpha_i = \varphi^* \alpha_i \in C^q_{d_1,\dots,d_q}$. For the other generators:

$$\begin{aligned}
\psi^* \beta^l &= \sum_{i=1}^{q} \sum_{j=0}^{d_i} \sum_{k=1}^{n} b^l_k(\varphi(y)) \frac{\partial \varphi^k}{\partial y_j * i}(y)\, dy^i_j - \kappa^l(y,v)\, dt \equiv \\
&\equiv \left(\sum_{i=1}^{q} \sum_{j=0}^{d_i} \sum_{k=1}^{n} y^i_{j+1} b^l_k(\varphi(y)) \frac{\partial \varphi^k}{\partial y_j * i}(y) \kappa^l(y,v) \right) dt \mod C^q_{d_1+1,\dots,d_q+1}, \quad l = 1, \dots, m
\end{aligned}$$

where the equivalence is thanks to (8.2). If we set $v^i := y_{d_i+1}$ and define κ in order to zero the coefficient dt, we get

$$\psi * I_f \subset C^q_{d_1+1,\dots,d_q+1}$$

We have put I into a controllable system. Therefore, $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is feedback linearizable. ∎

8.3.1 A sufficient condition

8.3.4 Theorem *A driftless system $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ with n states and m inputs is feedback linearizable if its derived coflag satisfies, at every point of a dense subset*

$$dim\ \Delta_k(x) = m + k, \quad k = 0, \dots, n-m$$

or equivalently if its derived flag satisfies dim $I^k(x) = n - m - k$ for $k = 0, \dots, n-m$.

Proof: We want to show that if the condition

$$\dim\ \Delta_k(x) = m + k, \quad k = 0, \dots, n-m$$

is satisfied then, the system $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is feedback linearizable. Using (8.3.3), this the same to prove that $I = \{f_1, \dots, f_m\}^\perp$ is linearizable.

Clearly, $s := \dim\ I = n - m$. The case $s = 0$ is trivial. We will focus on $s > 0$.

- $s = 1$
  By hypothesis

$$\begin{aligned}
\dim\ I = \dim I^{(0)} &= 1 \\
\dim\ I^{(1)} &= 0
\end{aligned}$$

hence, $I^{(0)}$ is spanned by one 1-form. Since the system is controllable, the Frobenius condition is not satisfied, i.e

$$d\alpha \wedge \alpha \neq 0$$

By (8.1.4), α can be written as

$$\alpha = dz^2 - z^1\, dz^3 + \sum_{i=4}^{m} a^i(z)\, dz^i$$

We get a submersion defined as

$$\varphi = (\varphi_1, \dots, \varphi_n) : \left(y_0^1, y_0^2, \dots, y_0^n\right) \longrightarrow (z_1, \dots, z_n)$$

such that $\varphi^* I^{(0)} \subset C_1^1$. If we set $\varphi_i(y_0^1, y_0^2, \dots, y_0^n) = y_0^i$ for $i = 2, \dots n$ and we choose φ_1 to zero the dt term we obtain:

$$\begin{aligned}
\varphi^*\alpha &= dy_0^2 - \varphi^1\, dy_0^3 + \sum_{i=4}^{n} (a^i \circ \varphi)\, dy_0^i \\
&\equiv \left(y_1^2 - y_1^3 \varphi^1 + \sum_{i=4}^{n} (a^i \circ \varphi) y_1^i \right) dt \mod C_1^1
\end{aligned}$$

which implies that $\varphi^*\alpha \in C_1^1$ and the system is linearizable.

- $s > 1$

  In this case, we use inductively the lemma (8.1.3) to find that

$$C(I^{(k)}) = s - k + 2, \quad k = 0, \dots, s-1$$

In particular,

$$\begin{aligned}
\dim\, I^{(0)} &= s + 2 \\
\dim\, I^{(s-1)} &= 3
\end{aligned}$$

Applying the theorem (8.1.2) there exists $(z_1, \dots, z_{s+2})$ such that $I^{(0)}$ can be expressed in this variables and $I^{(s-1)}$ can be expressed using only (z_1, z_2, z_3). Then, $I^{(s-1)}$ is a Pfaffian system of dimension 1 expressed in 3 variables. Clearly $I^{(s)} = \{0\}$, thus, we can apply the lemma (8.1.4) to express the 1-form belonging on $I^{(s-1)}$ as

$$\alpha_1 := dz^2 - z_3\, dz_1$$

From here we proceed by induction. Consider now $I^{(s-2)} = \{\alpha_1, \alpha_2\}$ where the expression of α_2 if found applying the Pfaff's theorem:

$$\alpha_2 := a^1(z)\,dz_1 + \sum_{i=3}^{n} a^i(z)\,dz_i$$

By definition of the successive derived flags, $d\alpha_1$ can be expressed in terms of α_1 and α_2, therefore

$$d\alpha_1 \wedge \alpha_1 \wedge \alpha_2 = 0 \Longrightarrow (-\,dz_3 \wedge dz_1) \wedge \left(dz^2 - z_3\,dz_1\right) \wedge \left(a^1(z)\,dz_1 + \sum_{i=3}^{n} a^i(z)\,dz_i\right) = 0$$

which implies that $a_4 = \cdots = a_n = 0$.

On the other hand,

$$d\alpha_2 \wedge \alpha_1 \wedge \alpha_2 \neq 0 \tag{8.4}$$

which implies a_1 and a_3 different from zero. We assume $a_3 = 1$. Since condition (8.4) is satisfied, this implies that a_1 does not depend only on z_1, z_2, z_3. We choose $a_1(z) = -z_4$. Therefore

$$\alpha_2 = dz_3 - z_4\,dz_1$$

Proceeding in the same way, we find

$$\begin{aligned} I^{(s-1)} &= \{\,dz_2 - z_3\,dz_1\} \\ I^{(s-2)} &= \{\,dz_2 - z_3\,dz_1,\, dz_3 - z_4\,dz_1\} \\ &\vdots \\ I^{(0)} &= \{\,dz_2 - z_3\,dz_1,\, dz_3 - z_4\,dz_1, \ldots,\, dz_{s+1} - z_{s+2}\,dz_1\} \end{aligned}$$

Now, we build a submersion

$$\varphi = (\varphi^1, \ldots, \varphi^{s+2}) : (y_0^1, \ldots, y_0^{s+2}) \longrightarrow (z_1, \ldots, z_{s+2})$$

which pulls back $I^{(0)}$ into $C^2_{s,s}$. Let's see this:

Recall that $I^{(0)}$ is a Pfaffian system that can be written using the z variables. So we can extend the submersion by adding

$$\xi^i := y_0^{i+2}, \quad i = 1, \ldots, n-s-2$$

to pull back $I^{(0)}$ into $C^{n-s}_{s,s,1,\ldots,1}$. As above, we set $\varphi^1 y = y_0^1$ and $\varphi^2(y) = y_0^2$ and construct $\varphi^3, \ldots, \varphi^{s+2}$ inductively. Since

$$\begin{aligned} \varphi^*(\,dz_2 - z^3\,dz^1) &= dy_0^2 - \varphi^3\,dy_0^1 \\ &\equiv (y_1^2 - y_1^1\varphi^3)\,dt \mod C^2_{1,1} \end{aligned}$$

we get a submersion $(\varphi^1, \varphi^2, \varphi^3)$ pulling back I^{s-1} into $C^2_{1,1}$ by setting $\varphi^3(y) := y^2_1/y^1_1$. Assume then that

$$(\varphi^1, \ldots, \varphi^{k+2}) : (y^1_0, \ldots, y^{k+2}_0) \longrightarrow (z_1, \ldots, z_{k+2})$$

is a subersion puling back I^{s-k} into $C^2_{k,k}$ and such that $\partial\varphi^{k+2}/\partial y^i_k = \partial\varphi^3/\partial y^i_1 \neq 0$. Now

$$\begin{aligned}\varphi^*(\,dz_{k+2} - z_{k+3}\,dz_1) &= d\varphi^{k+2} - \varphi^{k+3}\,dy^1_0 \\ &\equiv \left(\sum_{i=1}^{2}\sum_{j=0}^{k} y^i_{j+1}\frac{\partial\varphi^{k+2}}{\partial y^i_j} - y^1_1\varphi^{k+3}\right) dt \mod C^2_{k+1,k+1}\end{aligned}$$

and if we choose φ^{k+3} to zero the dt term, the map

$$(\varphi^1, \ldots, \varphi^{k+3}) : (y^1_0, \ldots, y^{k+3}_0) \longrightarrow (z_1, \ldots, z_{k+3})$$

is a submersion pulling back $I^{s-(k+1)}$ into $C^2_{k+1,k+1}$ and such that $\partial\varphi^{k+3}/\partial y^i_{k+1} = \partial\varphi^3/\partial y^i_1 \neq 0$. ■

8.3.5 Theorem *A driftless system $\dot{x} = f_1(x)u^1 + f_2(x)u^2$ with n states and two inputs is feedback linearizable if and only if its derived coflag satisfies, at every point of a dense open subset,*

$$dim\ \Delta_k = 2 + k, \quad k = 0, \ldots, n-2$$

or equivalently, if its derived flag satisfies dim $I^{(k)} = n - 2 - k$ for $k = 0, \ldots, n-2$.

Proof: Applying the theorem (8.3.4), there is proved an implication. It only remains to prove the necessity.

Consider a Pfaffian system $I = \{f_1, f_2\}^\perp$ be generated by $n-2$ independent 1-forms:

$$I = \{\alpha_1, \ldots, \alpha_s\}$$

Where $s := n-2$. The case $s = 1$ is trivial since $I^{(0)} = 1$ and $I^{(1)} = 0$. We focus on the case $s > 1$:

By assumption, I is linearizable, hence, controllable which implies that:

$$\begin{aligned}\dim C(I) &= s + 2 \\ \dim I^{(1)} &= s - 1\end{aligned}$$

Therefore, we can complete the basis $(\alpha_1, \ldots, \alpha_s)$ by adding α_{s+1} and α_{s+2}, hence, $(\alpha_1, \ldots, \alpha_{s+1})$ is a basis of $\Omega^1(X)$. We may assume that after reordering that $I^{(1)}$ is spanned by

$$I^{(1)} = \{\alpha_1, \ldots, \alpha_{s-1}\}$$

It is clear that

$$d\alpha_i \equiv 0 \mod I, \quad i = 1, \ldots, s-1$$
$$d\alpha_s \equiv \alpha_{s+1} \wedge \alpha_{s+2} \mod I$$

Thus,

$$d\alpha_i = \omega_1 \wedge \alpha_1 + \cdots + \omega_s \wedge \alpha_s \equiv \omega_s \wedge \alpha_s \mod I^(1)$$

for $\omega_1, \ldots, \omega_s \in \Omega^1(X)$. Hence,

$$d\alpha_1 \equiv \lambda_1^i \alpha_s \wedge \alpha_{s+1} + \lambda_2^i \alpha_s \wedge \alpha_{s+2} \mod I^{(1)}, \quad i = 1, \ldots, s-1$$

for a certain functions λ_1^i,λ_2^i, $\quad i = 1, \ldots, s-1$.

The second derived flag is defined as

$$I^{(2)} = \{\alpha \in I^{(1)} | \, d\alpha \equiv 0 \quad (I^{(1)})\}$$

therefore

$$\dim I^{(2)} = \dim I^{(1)} - \{\alpha | \, d\alpha \not\equiv 0 \quad (I^{(1)}) \text{ and linearly independents}\}$$

Setting $r := \{\alpha | \, d\alpha \not\equiv 0 \quad (I^{(1)})$ and linearly independents$\}$, the dimension of the second derived flag becomes

$$\dim I^{(2)} = s - 1 - r$$

for $r = 1, 2$ or 3. The condition of linearity independence is represented in terms of the functions λ_1^i, λ_2^i as the rank of the following matrix:

$$L = \begin{pmatrix} \lambda_1^1 & \cdots & \lambda_1^{s-1} \\ \lambda_2^1 & \cdots & \lambda_2^{s-1} \end{pmatrix}$$

Clearly, $r \neq 0$ because if $r = 0$, then $\dim I^{(1)} = \dim I^{(2)}$ and the system is not controllable. We only have to prove that $r = 1$. To do this we consider $q = 2$, where q is defined in definition (8.3.1). The same demonstration is applied to the case $q > 2$.

There exists two integers $d_1, d_2 \geq 0$ and a submersion

$$\varphi = (\varphi_1, \ldots, \varphi_{q+d_1+d_2}) : (y_0^1, \ldots, y_{d_1}^1, y_0^2, \ldots, y_{d_2}^2) \longrightarrow \{x\}$$

the system is linearizable if $\varphi^*(I) \subset C^2_{d_1,d_2}$. Let's see this:

The submersion has to depend on the variables $y_{d_1}^1$ and $y_{d_2}^1$, then, without loss of generality we assume

$$\frac{\partial \varphi}{\partial (y^1_{d_1}, y^1_{d_2})} \neq 0$$

For simplicity we denote the pullback of the ideal, form and function as

$$\begin{aligned} \widetilde{I} &= \varphi^*(I) \\ \widetilde{\alpha} &= \varphi^*(\alpha) \\ \widetilde{a} &= \varphi^* a \end{aligned}$$

We can assume that in fact, $\widetilde{I} \subset C^2_{r_1,r_2}$ where r_1 and r_2 are the smallest integers such that this is true and $r_1 \leq d_1$, $r_2 \leq d_2$. This means that each $\widetilde{\alpha}_i$ is linear combination of the basis elements of $C^2_{r_1,r_2}$ that is:

$$\widetilde{\alpha}_i = \sum_{j=1}^{r_1} b_1^{i,j} \omega^1_j + \sum_{k=1}^{r_2} b_2^{i,k} \omega^2_k, \quad i = 1, \ldots, s$$

where, at least, one of the functions b_1^{i,r_1} or b_2^{i,r_2} are not zero. We suppose that b_1^{s,r_1} or b_2^{s,r_2} are not zero.

Notice that the forms $\widetilde{\alpha}_{s+1}$ and $\widetilde{\alpha}_{s+2}$ do not belong on $\widetilde{I}$, so that we can still write them as

$$\alpha_{s+1} \equiv \sum_{j=1}^{d_1+1} b_1^{s+i,j} \omega_{j^1} + \sum_{k=1}^{d_2+1} b_2^{s+i,k} \omega^2_k \mod dt, \quad i = 1, 2$$

Since

$$\frac{\partial \varphi}{\partial (y^1_{d_1}, y^1_{d_2})} \neq 0$$

the square matrix

$$B = \begin{pmatrix} b_1^{s+1,d_1+1} & b_1^{s+2,d_1+1} \\ b_2^{s+1,d_2+1} & b_2^{s+2,d_2+1} \end{pmatrix}$$

is not the zero matrix.

Recall that we want $\widetilde{I} \subset C^2_{r_1,r_2}$. Since $d\alpha_i \equiv 0 \mod I$, for $i = 1, \ldots, s-1$, this implies that $d\widetilde{\alpha}_i \equiv 0 \mod C^2_{r_1,r_2}$, for $i = 1, \ldots, s-1$. Which implies that $b_1^{i,r_1} = b_2^{i,r_2} = 0, \quad i = 1, \ldots s-1$. Therefore $\widetilde{I}^{(1)} \subset C^2_{r_1-1,r_2-1}$.

The other condition, $d\alpha_s \equiv \alpha_{s+1} \wedge \alpha_{s+2} \mod I$ implies that

$$d\widetilde{\alpha}_s \equiv \widetilde{\alpha}_{s+1} \wedge \widetilde{\alpha}_{s+2} \mod (C^2_{r_1,r_2} \oplus \{dt\})$$

Using the fact that $\widetilde{I} \subset C^2_{r_1-1,r_2-1}$, we get

$$d\widetilde{\alpha}_s \equiv \widetilde{\lambda}_1^1 \widetilde{\alpha}_s \wedge \widetilde{\alpha}_{s+1} + \widetilde{\lambda}_2^1 \widetilde{\alpha}_s \wedge \widetilde{\alpha}_{s+2} \mod C^2_{r_1-1,r_2-1}, \quad i = 1, \dots s-1$$

On the other hand, we also have

$$d\widetilde{\alpha}_i \equiv 0 \mod (C^2_{r_1-1,r_2-1} \oplus \{dt\}), \quad i = 1, \dots, s-1$$

We know that the expression $\widetilde{\lambda}_1^1 \widetilde{\alpha}_s \wedge \widetilde{\alpha}_{s+1} + \widetilde{\lambda}_2^1 \widetilde{\alpha}_s \wedge \widetilde{\alpha}_{s+2}$ contains a linear combination of four decomposable 2-forms as:

$$\begin{aligned}
&b_1^{s,r_1} \left(\widetilde{\lambda}_1^i b_1^{s+1,d_1+1} + \widetilde{\lambda}_2^i b_1^{s+2,d_1+1} \right) \omega_{r_1}^1 \wedge \omega_{d_1+1}^1 \\
&\quad + b_1^{s,r_1} \left(\widetilde{\lambda}_1^i b_2^{s+1,d_2+1} + \widetilde{\lambda}_2^i b_2^{s+2,d_2+1} \right) \omega_{r_1}^1 \wedge \omega_{d_2+1}^2 \\
&\quad + b_2^{s,r_2} \left(\widetilde{\lambda}_1^i b_1^{s+1,d_1+1} + \widetilde{\lambda}_2^i b_1^{s+2,d_1+1} \right) \omega_{r_2}^2 \wedge \omega_{d_1+1}^1 \\
&\quad + b_2^{s,r_2} \left(\widetilde{\lambda}_1^i b_2^{s+1,d_2+1} + \widetilde{\lambda}_2^i b_2^{s+2,d_2+1} \right) \omega_{r_2}^2 \wedge \omega_{d_2+1}^2
\end{aligned}$$

that are independent mod $C^2_{r_1-1,r_2-1} \oplus \{dt\}$. Since $b_1^{s,r_1} \neq 0$ or $b_2^{s,r_2} \neq 0$, we deduce that

$$\begin{pmatrix} b_1^{s+1,d_1+1} & b_1^{s+2,d_1+1} \\ b_2^{s+1,d_2+1} & b_2^{s+2,d_2+1} \end{pmatrix} \begin{pmatrix} \widetilde{\lambda}_1^1 & \cdots & \widetilde{\lambda}_1^{s-1} \\ \widetilde{\lambda}_2^1 & \cdots & \widetilde{\lambda}_2^{s-1} \end{pmatrix} = 0$$

Since the matrix $B \neq 0$, the rank of $\widetilde{L}$ which is equal to the rank of L is not maximal. Hence the dim $I^{(2)} = s-2$.

Therefore, using lemma (8.1.3), we get

$$\dim C(I^{(1)}) = s+1$$

Applying the theorem (8.1.2), we obtain that $I^{(1)}$ is a system of dimension $s-1$ in $s+1$ variables. And $I^{(1)}$ is included under submersion into a linearizable system which implies that $I^{(1)}$ is linearizable itself, hence we are left with exactly the same problem, but now with a system of dimension $s-1$ instead of s. Proceeding by induction till the dimension 1 we obtain the necessary condition

$$\dim I^{(k)} = s-k, \quad k = 1, \dots s$$

■

8.4 Application to a Kinematic Car

Consider the equations of motion for a kinematic car described in [13], represented in figure (8.1):

$$\begin{cases} \dot{x} = \cos\theta\cos\phi v_1 \\ \dot{y} = \sin\theta\cos\phi v_1 \\ \dot{\theta} = \sin\phi v_1 \\ \dot{\phi} = v_2 \end{cases}$$

where (x, y) is the position of the car, θ is the angle between the horizontal and the car, ϕ is the steering angle, v_1, the fordward velocity and v_2 the steering angle velocity. For simplicity, the length of the car is assumed to be 1.

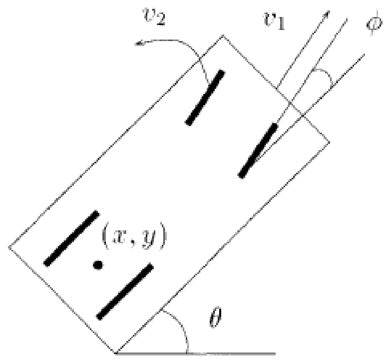

Figure 8.1: The kinematic car

Our goal is to show that the system is feedback linearizable applying the theorem (8.3.5) and finding the flat outputs using the constructive demonstration of the theorem (8.3.4).

So, in this case, we want to see that

$$\begin{aligned} \dim\, I^{(0)} &= 2 \\ \dim\, I^{(1)} &= 1 \\ \dim\, I^{(2)} &= 0 \end{aligned}$$

The vector fields of the system are:

$$g_1 = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

To find the generators of $I^{(0)}$ we have to seek for α_1 and α_2 such that the contraction with the vector fields being zero.

Consider

$$
\begin{aligned}
\alpha_1 &= a_1\, dx + b_1\, dy + c_1\, d\theta + d_1\, d\phi \\
\alpha_2 &= a_2\, dx + b_2\, dy + c_2\, d\theta + d_2\, d\phi
\end{aligned}
$$

Then, the following relations has to be satisfied:

$$
\begin{aligned}
a_i \cos\theta \cos\phi + b_i \sin\theta \cos\phi + c_i \sin\phi = 0 \\
d_i = 0
\end{aligned}
$$

for $i = 1, 2$. Therefore, the two 1-forms could be

$$
\begin{aligned}
\alpha_1 &= -\sin\theta\, dx + \cos\theta\, dy \\
\alpha_2 &= \cos\theta\, dx + \sin\theta\, dy - \frac{\cos\phi}{\sin\phi}\, d\theta
\end{aligned}
$$

We calculate its exterior derivative to define the successive derived flags:

$$
\begin{aligned}
d\alpha_1 &= -\cos\theta\, d\theta \wedge\, dx - \sin\theta\, d\theta \wedge\, dy \\
d\alpha_2 &= -\sin\theta\, d\theta \wedge\, dx + \cos\theta\, d\theta \wedge\, dy + \frac{1}{\sin^2\phi}\, d\phi \wedge\, d\theta
\end{aligned}
$$

Clearly

$$
\begin{aligned}
& d\alpha_1 = -\, d\theta \wedge \alpha_2 \\
& d\alpha_2 \neq A \wedge \alpha_1 + B \wedge \alpha_2, \quad \text{for any 1-form} A, B
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I^{(0)} &= \{\alpha_1, \alpha_2\} \\
I^{(1)} &= \{\alpha_1\} \\
I^{(2)} &= \{0\}
\end{aligned}
$$

$$
\begin{aligned}
\dim\, I^{(0)} &= 2\\
\dim\, I^{(1)} &= 1\\
\dim\, I^{(2)} &= 0
\end{aligned}
$$

By theorem (8.3.5) the system is feedback linearizable i.e there exists a submersion φ such that $\varphi^* I \subset C^2_{d_1,d_2}$. Let's first find the generators in a normal form.

We apply the Engel's theorem to find

$$I = \{\, dz_2 - z_3\, dz_1,\, dz_3 - z_4\, dz_1\}$$

Setting

$$
\begin{aligned}
z_1 &= \theta\\
z_2 &= -x\sin\theta + y\cos\theta\\
z_3 &= x\cos\theta + y\sin\theta\\
z_4 &= -x\sin\theta + y\cos\theta - \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

The 1-forms can be written in the new variables as

$$
\begin{aligned}
\alpha_2 &= dz_2 - z_3\, dz_1\\
\alpha_1 &= dz_3 - z_4\, dz_1
\end{aligned}
$$

Clearly, the flat outputs are $y^1_0 = z_1$ and $y^2_0 = z_2$. Let's see that, the same flat outputs are obtained by putting the system into controllable linear one.

The system (8.4) can be put into a Brunovsky normal form by a prolongation and an adjustment of the Pfaffian system basis as follows

$$C^2_{3,3} = \{\omega^1_1, \omega^1_2, \omega^1_3, \omega^1_2, \omega^2_2, \omega^2_3\}$$

But we will restrict our computations to

$$C^2_{2,2} = \{\omega^1_1, \omega^1_2, \omega^1_2, \omega^2_2\}$$

where

$$\omega^i_j = dy^1_{j-1} - y^i_j\, dt, \quad i, j = 1, 2$$

We can construct a submersion φ such that

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) : (y_0^1, y_0^2, y_0^3, y_0^4) \longrightarrow (z_1, z_2, z_3, z_4)$$

We set $\varphi_1(y) = y_0^1$ and $\varphi_2(y) = y_0^2$ and we look for $\varphi_3(y)$ such that vanishes the dt term:

$$\begin{aligned}
\varphi^*(\,dz_2 - z_3\,dz_1) &= \,dy_0^2 - \varphi_3\,dy_0^1 \\
&= \omega_1^2 + y_1^2\,dt - \varphi(\omega_1^1 + y_1^1\,dt) \\
&\equiv (y_1^2 - \varphi_3 y_1^1)\,dt \mod C_{3,3}^2
\end{aligned}$$

Therefore, $\varphi^*(I^{(1)}) \subset C_{2,2}^2$ if we choose $\varphi_3(y) = y_1^2/y_1^1$ or, in other words:

$$z_3 = \frac{y_1^2}{y_1^1} = \frac{\dot{y}_0^1}{\dot{y}_0^2}$$

Let's determine $\varphi_4(y)$ such that $\varphi^*(I^{(0)}) \subset C_{2,2}^2$:

$$\begin{aligned}
\varphi^*(\,dz_3 - z_4\,dz_1) &= \,d\varphi_3 - \varphi_4\,d\varphi_1 \\
&= \,dy_1^2\left(\frac{1}{y_1^1}\right) - \,dy_1^1\left(\frac{y_1^2}{(y_1^1)^2}\right) - \varphi_4\,dy_0^1 \\
&= (\omega_2^2 + y_2^2\,dt)\left(\frac{1}{y_1^1}\right) - (\omega_2^1 + y_2^1\,dt)\left(\frac{y_1^2}{(y_1^1)^2}\right) - \varphi_4(\omega_1^1 + y_1^1\,dt) \\
&\equiv \left(\frac{y_2^2}{y_1^1} - \frac{y_2^1 y_1^2}{(y_1^1)^2} - \varphi_4 y_1^1\right)dt \mod C_{2,2}^2
\end{aligned}$$

Therefore

$$\varphi_4 = \frac{y_2^2 y_1^1 - y_2^1 y_1^2}{(y_1^1)^3}$$

or, in other words

$$z_4 = \frac{\ddot{y}_0^2 \dot{y}_0^1 - \ddot{y}_0^1 \dot{y}_0^2}{(\dot{y}_0^1)^3}$$

We have expressed all the variables in terms of $y_0^1 = z_1$, $y_0^2 = z_2$, thus, these are the flat outputs of the system.

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